Journal of Nonlinear Mathematical Physics
Vol. **28(3)**; September (2021), *pp.* 277–291
DOI: https://doi.org/10.2991/jnmp.k.210203.001; ISSN 1402-9251; eISSN 1776-0852
https://www.atlantis-press.com/journals/jnmp

Research Article

Affine Ricci Solitons of Three-Dimensional Lorentzian Lie Groups

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ARTICLE INFO

Article History

Received 28 December 2020
Accepted 28 January 2021

Keywords

Canonical connections
Kobayashi-Nomizu connections
perturbed canonical connections
perturbed Kobayashi-Nomizu connections
affine Ricci solitons
three-dimensional Lorentzian Lie groups

2010 Mathematics Subject Classification

53C40
53C42

ABSTRACT

In this paper, we classify affine Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections and perturbed canonical connections and perturbed Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups with some product structure.

1. INTRODUCTION

The concept of the Ricci soliton is introduced by Hamilton in [9], which is a natural generalization of Einstein metrics. Study of Ricci soliton over different geometric spaces is one of interesting topics in geometry and mathematical physics. In particular, it has become more important after G. Perelman applied Ricci solitons to solve the long standing Poincare conjecture. In [10,13,15–18], Einstein manifolds associated to affine connections (especially semi-symmetric metric connections and semi-symmetric non-metric connections) were studied (see the definition 3.2 in [18] and the definition 3.1 in [10]). It is natural to study Ricci solitons associated to affine connections. Affine Ricci solitons had been introduced and studied, for example, see [6,8,11,12,14].

Our motivation is to find more examples of affine Ricci solitons. A three-dimensional Lie group $G_i(i = 1, \cdots, 7)$ is a sub-Riemannian manifold. In [1], Balogh, Tyson and Vecchi applied a Riemannian approximation scheme to get a Gauss-Bonnet theorem in the Heisenberg group $\mathbb{H}^3$. Let $T\mathbb{H}^3 = \mathrm{span}\{e_1, e_2, e_3\}$, then they took the distribution $H = \mathrm{span}\{e_1, e_2\}$ and $H^\perp = \mathrm{span}\{e_3\}$ (for details, see [1]). Similarly in [20], for the affine group and the group of rigid motions of the Minkowski plane, we took the similar distributions. In [21], for the Lorentzian Heisenberg group, we also took the similar construction. Motivated by [1,20,21], we consider the similar distribution $H = \mathrm{span}\{e_1, e_2\}$ and $H^\perp = \mathrm{span}\{e_3\}$ for the three dimensional Lorentzian Lie group $G_i(i = 1, \cdots, 7)$. Then for the above distribution, we have a natural product structure J: $Je_1 = e_1$, $Je_2 = e_2$, $Je_3 = -e_3$. In [7], Etayo and Santamaria studied some affine connections on manifolds with the product structure or the complex structure. In particular, the canonical connection and the Kobayashi-Nomizu connection for a product structure were studied. So we consider the canonical connection and the Kobayashi-Nomizu connection associated to the above distribution on the G_i and get affine Ricci solitons associated to the canonical connection and the Kobayashi-Nomizu connection. In particular, from our results, we can get affine Einstein manifolds associated to the canonical connection and the Kobayashi-Nomizu connection. It is interesting to consider relations between affine Ricci solitons associated to the canonical connection and the Kobayashi-Nomizu connection and Ricci solitons associated to the Levi-Civita connection. It is also interesting to study affine Ricci solitons associated to other affine connections, for example, Schouten-Van Kampen connections and Vranceanu connections associated to the above product structure and semi-symmetric connections.

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Data availability statement: The authors confirm that the data supporting the findings of this study are available within the article.

By the canonical connection and the Kobayashi-Nomizu connection on three-dimensional Lorentzian Lie groups, we obtain some examples of affine Ricci solitons. But we find that the coefficient λ of the metric tensor g in the Ricci soliton equation (see (3.13) and (3.14)) is always zero for these obtained examples. In order to obtain more interesting examples with the non zero coefficient λ, we introduce perturbed canonical connections and perturbed Kobayashi-Nomizu connections in Section 4. Using these perturbed connections, we get some examples of affine Ricci solitons with the non zero coefficient λ.

In [3], Calvaruso studied three-dimensional generalized Ricci solitons, both in Riemannian and Lorentzian settings. He determined their homogeneous models, classifying left-invariant generalized Ricci solitons on three-dimensional Lie groups. Then it is natural to classify affine Ricci solitons on three-dimensional Lie groups. In [19], we introduced a particular product structure on three-dimensional Lorentzian Lie groups and we computed canonical connections and Kobayashi-Nomizu connections and their curvature on three-dimensional Lorentzian Lie groups with this product structure. We defined algebraic Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections. We classified algebraic Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups with this product structure. In this paper, we classify affine Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections and perturbed canonical connections and perturbed Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups with this product structure.

In Section 2, we recall the classification of three-dimensional Lorentzian Lie groups. In Section 3, we classify affine Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups with this product structure. In Section 4, we classify affine Ricci solitons associated to perturbed canonical connections and perturbed Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups with this product structure.

2. THREE-DIMENSIONAL LORENTZIAN LIE GROUPS

In this section, we recall the classification of three-dimensional Lorentzian Lie groups in [4,5](also see Theorems 2.1 and 2.2 in [2]).

Theorem 2.1. *Let (G, g) be a three-dimensional connected unimodular Lie group, equipped with a left-invariant Lorentzian metric. Then there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G is one of the following:*

($\mathfrak{g}_1$):

$$[e_1, e_2] = \alpha e_1 - \beta e_3, \quad [e_1, e_3] = -\alpha e_1 - \beta e_2, \quad [e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \quad \alpha \neq 0. \tag{2.1}$$

($\mathfrak{g}_2$):

$$[e_1, e_2] = \gamma e_2 - \beta e_3, \quad [e_1, e_3] = -\beta e_2 - \gamma e_3, \quad [e_2, e_3] = \alpha e_1, \quad \gamma \neq 0. \tag{2.2}$$

($\mathfrak{g}_3$):

$$[e_1, e_2] = -\gamma e_3, \quad [e_1, e_3] = -\beta e_2, \quad [e_2, e_3] = \alpha e_1. \tag{2.3}$$

($\mathfrak{g}_4$):

$$[e_1, e_2] = -e_2 + (2\eta - \beta) e_3, \quad \eta = 1 \text{ or } -1, \quad [e_1, e_3] = -\beta e_2 + e_3, \quad [e_2, e_3] = \alpha e_1. \tag{2.4}$$

Theorem 2.2. *Let (G, g) be a three-dimensional connected non-unimodular Lie group, equipped with a left-invariant Lorentzian metric. Then there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G is one of the following:*

($\mathfrak{g}_5$):

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \alpha e_1 + \beta e_2, \quad [e_2, e_3] = \gamma e_1 + \delta e_2, \quad \alpha + \delta \neq 0, \quad \alpha\gamma + \beta\delta = 0. \tag{2.5}$$

($\mathfrak{g}_6$):

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \quad [e_2, e_3] = 0, \quad \alpha + \delta \neq 0, \quad \alpha\gamma - \beta\delta = 0. \tag{2.6}$$

($\mathfrak{g}_7$):

$$[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3, \quad [e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3, \quad [e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3, \quad \alpha + \delta \neq 0, \quad \alpha\gamma = 0. \tag{2.7}$$

3. AFFINE RICCI SOLITONS ASSOCIATED TO CANONICAL CONNECTIONS AND KOBAYASHI-NOMIZU CONNECTIONS ON THREE-DIMENSIONAL LORENTZIAN LIE GROUPS

Throughout this paper, we shall by $\{G_i\}_{i=1,\dots,7}$, denote the connected, simply connected three-dimensional Lie group equipped with a left-invariant Lorentzian metric g and having Lie algebra $\{\mathfrak{g}\}_{i=1,\dots,7}$. Let ∇ be the Levi-Civita connection of G_i and R its curvature tensor, taken with the convention

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{3.1}$$

The Ricci tensor of (G_i, g) is defined by

$$\rho(X,Y) = -g(R(X,e_1)Y,e_1) - g(R(X,e_2)Y,e_2) + g(R(X,e_3)Y,e_3), \tag{3.2}$$

where $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis, with e_3 timelike. We define a product structure J on G_i by

$$Je_1 = e_1,\ Je_2 = e_2,\ Je_3 = -e_3, \tag{3.3}$$

then $J^2 = \mathrm{id}$ and $g(Je_j, Je_j) = g(e_j, e_j)$. By [7], we define the canonical connection and the Kobayashi-Nomizu connection as follows:

$$\nabla^0_X Y = \nabla_X Y - \frac{1}{2}(\nabla_X J)JY, \tag{3.4}$$

$$\nabla^1_X Y = \nabla^0_X Y - \frac{1}{4}[(\nabla_Y J)JX - (\nabla_{JY}J)X]. \tag{3.5}$$

We define

$$R^0(X,Y)Z = \nabla^0_X\nabla^0_Y Z - \nabla^0_Y\nabla^0_X Z - \nabla^0_{[X,Y]}Z, \tag{3.6}$$

$$R^1(X,Y)Z = \nabla^1_X\nabla^1_Y Z - \nabla^1_Y\nabla^1_X Z - \nabla^1_{[X,Y]}Z. \tag{3.7}$$

The Ricci tensors of (G_i, g) associated to the canonical connection and the Kobayashi-Nomizu connection are defined by

$$\rho^0(X,Y) = -g(R^0(X,e_1)Y,e_1) - g(R^0(X,e_2)Y,e_2) + g(R^0(X,e_3)Y,e_3), \tag{3.8}$$

$$\rho^1(X,Y) = -g(R^1(X,e_1)Y,e_1) - g(R^1(X,e_2)Y,e_2) + g(R^1(X,e_3)Y,e_3). \tag{3.9}$$

Let

$$\tilde{\rho}^0(X,Y) = \frac{\rho^0(X,Y) + \rho^0(Y,X)}{2}, \tag{3.10}$$

and

$$\tilde{\rho}^1(X,Y) = \frac{\rho^1(X,Y) + \rho^1(Y,X)}{2}. \tag{3.11}$$

Since $(L_V g)(Y,Z) := g(\nabla_Y V, Z) + g(Y, \nabla_Z V)$, we let

$$(L^j_V g)(Y,Z) := g(\nabla^j_Y V, Z) + g(Y, \nabla^j_Z V), \tag{3.12}$$

for $j = 0, 1$ and vector fields V, Y, Z.

Definition 3.1. *(G_i, g, J) is called the affine Ricci soliton associated to the connection ∇^0 if it satisfies*

$$(L^0_V g)(Y,Z) + 2\tilde{\rho}^0(Y,Z) + 2\lambda g(Y,Z) = 0 \tag{3.13}$$

where λ is a real number and $V = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ and $\lambda_1, \lambda_2, \lambda_3$ are real numbers. (G_i, g, J) is called the affine Ricci soliton associated to the connection ∇^1 if it satisfies

$$(L^1_V g)(Y,Z) + 2\tilde{\rho}^1(Y,Z) + 2\lambda g(Y,Z) = 0 \tag{3.14}$$

By (2.25) in [19], we have for (G_1, g, J, ∇^0)

$$\tilde{\rho}^0(e_1,e_1) = -\left(\alpha^2 + \frac{\beta^2}{2}\right), \quad \tilde{\rho}^0(e_1,e_2) = 0, \tag{3.15}$$
$$\tilde{\rho}^0(e_1,e_3) = \frac{\alpha\beta}{4}, \quad \tilde{\rho}^0(e_2,e_2) = -\left(\alpha^2 + \frac{\beta^2}{2}\right),$$
$$\tilde{\rho}^0(e_2,e_3) = \frac{\alpha^2}{2}, \quad \tilde{\rho}^0(e_3,e_3) = 0.$$

By Lemma 2.4 in [19] and (3.12), we have for (G_1, g, J, ∇^0, V)

$$(L^0_V g)(e_1,e_1) = 2\lambda_2\alpha, \qquad (L^0_V g)(e_1,e_2) = -\lambda_1\alpha \tag{3.16}$$

$$(L^0_V g)(e_1,e_3) = -\frac{\beta}{2}\lambda_2, \quad (L^0_V g)(e_2,e_2) = 0,$$
$$(L^0_V g)(e_2,e_3) = \frac{\beta}{2}\lambda_1, \quad (L^0_V g)(e_3,e_3) = 0.$$

If (G_1, g, J, V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases} 2\lambda_2\alpha - 2\alpha^2 - \beta^2 + 2\lambda = 0, \\ \lambda_1\alpha = 0, \\ -\beta\lambda_2 + \alpha\beta = 0, \\ -2\alpha^2 - \beta^2 + 2\lambda = 0, \\ \frac{\beta}{2}\lambda_1 + \alpha^2 = 0, \\ \lambda = 0. \end{cases} \tag{3.17}$$

Solve (3.17), we have

Theorem 3.2. *(G_1, g, J, V) is not an affine Ricci soliton associated to the connection ∇^0.*

By (2.33) in [19], we have for (G_1, g, J, ∇^1)

$$\begin{aligned} \tilde{\rho}^1(e_1,e_1) &= -\left(\alpha^2+\beta^2\right), & \tilde{\rho}^1(e_1,e_2) &= \alpha\beta, \\ \tilde{\rho}^1(e_1,e_3) &= -\frac{\alpha\beta}{2}, & \tilde{\rho}^1(e_2,e_2) &= -\left(\alpha^2+\beta^2\right), \\ \tilde{\rho}^1(e_2,e_3) &= \frac{\alpha^2}{2}, & \tilde{\rho}^1(e_3,e_3) &= 0. \end{aligned} \tag{3.18}$$

By Lemma 2.8 in [19] and (3.12), we have for (G_1, g, J, ∇^1, V)

$$\begin{aligned} (L^1_V g)(e_1,e_1) &= 2\lambda_2\alpha, & (L^1_V g)(e_1,e_2) &= -\lambda_1\alpha, \\ (L^1_V g)(e_1,e_3) &= \lambda_1\alpha - \beta\lambda_2, & (L^1_V g)(e_2,e_2) &= 0, \\ (L^1_V g)(e_2,e_3) &= \beta\lambda_1 - \alpha\lambda_2 - \alpha\lambda_3, & (L^1_V g)(e_3,e_3) &= 0. \end{aligned} \tag{3.19}$$

If (G_1, g, J, V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases} \lambda_2\alpha - \alpha^2 - \beta^2 + \lambda = 0, \\ -\lambda_1\alpha + 2\alpha\beta = 0, \\ \lambda_1\alpha - \beta\lambda_2 - \alpha\beta = 0, \\ -\alpha^2 - \beta^2 + \lambda = 0, \\ \beta\lambda_1 - \alpha\lambda_2 - \alpha\lambda_3 + \alpha^2 = 0, \\ \lambda = 0. \end{cases} \tag{3.20}$$

Solve (3.20), we have

Theorem 3.3. *(G_1, g, J, V) is not an affine Ricci soliton associated to the connection ∇^1.*

By (2.44) in [19], we have for (G_2, g, J, ∇^0)

$$\begin{aligned} \tilde{\rho}^0(e_1,e_1) &= -\left(\gamma^2+\frac{\alpha\beta}{2}\right), & \tilde{\rho}^0(e_1,e_2) &= 0, \\ \tilde{\rho}^0(e_1,e_3) &= 0, & \tilde{\rho}^0(e_2,e_2) &= -\left(\gamma^2+\frac{\alpha\beta}{2}\right), \\ \tilde{\rho}^0(e_2,e_3) &= \frac{\beta\gamma}{2} - \frac{\alpha\gamma}{4}, & \tilde{\rho}^0(e_3,e_3) &= 0. \end{aligned} \tag{3.21}$$

By Lemma 2.14 in [19] and (3.12), we have for (G_2, g, J, ∇^0, V)

$$\begin{aligned} (L^0_V g)(e_1,e_1) &= 0, & (L^0_V g)(e_1,e_2) &= \lambda_2\gamma \\ (L^0_V g)(e_1,e_3) &= -\frac{\alpha}{2}\lambda_2, & (L^0_V g)(e_2,e_2) &= -2\gamma\lambda_1, \\ (L^0_V g)(e_2,e_3) &= \frac{\alpha}{2}\lambda_1, & (L^0_V g)(e_3,e_3) &= 0. \end{aligned} \tag{3.22}$$

If (G_2, g, J, V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases} -\left(\gamma^2+\frac{\alpha\beta}{2}\right)+\lambda=0, \\ \lambda_2\gamma=0, \\ \alpha\lambda_2=0, \\ -\gamma\lambda_1-\left(\gamma^2+\frac{\alpha\beta}{2}\right)+\lambda=0, \\ \frac{\alpha}{2}\lambda_1+2\left(\frac{\beta\gamma}{2}-\frac{\alpha\gamma}{4}\right)=0, \\ \lambda=0. \end{cases} \tag{3.23}$$

Solve (3.23), we have

Theorem 3.4. *(G_2, g, J, V) is not an affine Ricci soliton associated to the connection ∇^0.*

By (2.54) in [19], we have for (G_2, g, J, ∇^1)

$$\begin{aligned} &\tilde{\rho}^1(e_1,e_1)=-\left(\beta^2+\gamma^2\right), && \tilde{\rho}^1(e_1,e_2)=0, \\ &\tilde{\rho}^1(e_1,e_3)=0, && \tilde{\rho}^1(e_2,e_2)=-\left(\gamma^2+\alpha\beta\right), \\ &\tilde{\rho}^1(e_2,e_3)=-\frac{\alpha\gamma}{2}, && \tilde{\rho}^1(e_3,e_3)=0. \end{aligned} \tag{3.24}$$

By Lemma 2.18 in [19] and (3.12), we have for (G_2, g, J, ∇^1, V)

$$\begin{aligned} &(L^1_V g)(e_1,e_1)=0, && (L^1_V g)(e_1,e_2)=\lambda_2\gamma, \\ &(L^1_V g)(e_1,e_3)=-\alpha\lambda_2+\gamma\lambda_3, && (L^1_V g)(e_2,e_2)=-2\gamma\lambda_1, \\ &(L^1_V g)(e_2,e_3)=\lambda_1\beta, && (L^1_V g)(e_3,e_3)=0. \end{aligned} \tag{3.25}$$

If (G_2, g, J, V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases} -\beta^2-\gamma^2+\lambda=0, \\ \lambda_2\gamma=0, \\ -\alpha\lambda_2+\gamma\lambda_3=0, \\ -\gamma\lambda_1-\left(\gamma^2+\alpha\beta\right)+\lambda=0, \\ \lambda_1\beta-\alpha\gamma=0, \\ \lambda=0. \end{cases} \tag{3.26}$$

Solve (3.26), we have

Theorem 3.5. *(G_2, g, J, V) is not an affine Ricci soliton associated to the connection ∇^1.*

By (2.64) in [19], we have for (G_3, g, J, ∇^0)

$$\begin{aligned} &\tilde{\rho}^0(e_1,e_1)=-\gamma a_3, && \tilde{\rho}^0(e_1,e_2)=0, \\ &\tilde{\rho}^0(e_1,e_3)=0, && \tilde{\rho}^0(e_2,e_2)=-\gamma a_3, \\ &\tilde{\rho}^0(e_2,e_3)=0, && \tilde{\rho}^0(e_3,e_3)=0, \end{aligned} \tag{3.27}$$

where $a_3=\frac{1}{2}(\alpha+\beta-\gamma)$. By Lemma 2.24 in [19] and (3.12), we have for (G_3, g, J, ∇^0, V)

$$\begin{aligned} &(L^0_V g)(e_1,e_1)=0, && (L^0_V g)(e_1,e_2)=0, \\ &(L^0_V g)(e_1,e_3)=-a_3\lambda_2, && (L^0_V g)(e_2,e_2)=0, \\ &(L^0_V g)(e_2,e_3)=a_3\lambda_1, && (L^0_V g)(e_3,e_3)=0. \end{aligned} \tag{3.28}$$

If (G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases} \gamma a_3=0, \\ \lambda_2 a_3=0, \\ \lambda_1 a_3=0, \\ \lambda=0. \end{cases} \tag{3.29}$$

Solve (3.29), we have

Theorem 3.6. *(G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^0 if and only if*

(i) $\lambda = 0, \alpha + \beta - \gamma = 0,$
(ii) $\lambda = 0, \alpha + \beta - \gamma \neq 0, \gamma = \lambda_1 = \lambda_2 = 0.$

By (2.69) in [19], we have for (G_3, g, J, ∇^1)

$$\begin{aligned} &\widetilde{\rho}^1(e_1,e_1) = \gamma(a_1 - a_3), && \widetilde{\rho}^1(e_1,e_2) = 0, \\ &\widetilde{\rho}^1(e_1,e_3) = 0, && \widetilde{\rho}^1(e_2,e_2) = -\gamma(a_2 + a_3), \\ &\widetilde{\rho}^1(e_2,e_3) = 0, && \widetilde{\rho}^1(e_3,e_3) = 0, \end{aligned} \tag{3.30}$$

where $a_1 = \frac{1}{2}(\alpha - \beta - \gamma),\ a_2 = \frac{1}{2}(\alpha - \beta + \gamma)$. By Lemma 2.27 in [19] and (3.12), we have for (G_3, g, J, ∇^1, V)

$$\begin{aligned} &(L^1_V g)(e_1,e_1) = 0, && (L^1_V g)(e_1,e_2) = 0, \\ &(L^1_V g)(e_1,e_3) = -(a_2 + a_3)\lambda_2, && (L^1_V g)(e_2,e_2) = 0, \\ &(L^1_V g)(e_2,e_3) = \lambda_1(a_3 - a_1), && (L^1_V g)(e_3,e_3) = 0. \end{aligned} \tag{3.31}$$

If (G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases} \gamma(a_1 - a_3) + \lambda = 0, \\ (a_2 + a_3)\lambda_2 = 0, \\ -\gamma(a_2 + a_3) + \lambda = 0, \\ \lambda_1(a_3 - a_1) = 0, \\ \lambda = 0. \end{cases} \tag{3.32}$$

Solve (3.32), we have

Theorem 3.7. *(G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^1 if and only if the following statements hold true*

(i) $\lambda = 0, \gamma \neq 0, \alpha = \beta = 0,$
(ii) $\lambda = 0, \gamma = 0, \alpha\lambda_2 = 0, \lambda_1\beta = 0.$

By (2.81) in [19], we have for (G_4, g, J, ∇^0)

$$\begin{aligned} &\widetilde{\rho}^0(e_1,e_1) = (2\eta - \beta)b_3 - 1, && \widetilde{\rho}^0(e_1,e_2) = 0, \\ &\widetilde{\rho}^0(e_1,e_3) = 0, && \widetilde{\rho}^0(e_2,e_2) = (2\eta - \beta)b_3 - 1, \\ &\widetilde{\rho}^0(e_2,e_3) = \frac{b_3 - \beta}{2}, && \widetilde{\rho}^0(e_3,e_3) = 0, \end{aligned} \tag{3.33}$$

where $b_3 = \frac{\alpha}{2} + \eta$. By Lemma 2.32 in [19] and (3.12), we have for (G_4, g, J, ∇^0, V)

$$\begin{aligned} &(L^0_V g)(e_1,e_1) = 0, && (L^0_V g)(e_1,e_2) = -\lambda_2, \\ &(L^0_V g)(e_1,e_3) = -b_3\lambda_2, && (L^0_V g)(e_2,e_2) = 2\lambda_1, \\ &(L^0_V g)(e_2,e_3) = b_3\lambda_1, && (L^0_V g)(e_3,e_3) = 0. \end{aligned} \tag{3.34}$$

If (G_4, g, J, V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases} (2\eta - \beta)b_3 - 1 + \lambda = 0, \\ \lambda_2 = 0, \\ \lambda_1 + (2\eta - \beta)b_3 - 1 + \lambda = 0, \\ \lambda_1 b_3 + b_3 - \beta = 0, \\ \lambda = 0. \end{cases} \tag{3.35}$$

Solve (3.35), we have

Theorem 3.8. *(G_4, g, J, V) is an affine Ricci soliton associated to the connection ∇^0 if and only if $\lambda = \lambda_1 = \lambda_2 = 0, \alpha = 0, \beta = \eta$.*

By (2.89) in [19], we have for (G_4, g, J, ∇^1)

$$\begin{aligned} &\widetilde{\rho}^1(e_1,e_1) = -[1 + (\beta - 2\eta)(b_3 - b_1)], && \widetilde{\rho}^1(e_1,e_2) = 0, \\ &\widetilde{\rho}^1(e_1,e_3) = 0, && \widetilde{\rho}^1(e_2,e_2) = -[1 + (\beta - 2\eta)(b_2 + b_3)], \\ &\widetilde{\rho}^1(e_2,e_3) = \frac{\alpha + b_3 - b_1 - \beta}{2}, && \widetilde{\rho}^1(e_3,e_3) = 0, \end{aligned} \tag{3.36}$$

where $b_1 = \frac{\alpha}{2} + \eta - \beta$, $b_2 = \frac{\alpha}{2} - \eta$. By Lemma 2.36 in [19] and (3.12), we have for (G_4, g, J, ∇^1, V)

$$\begin{aligned} &(L^1_V g)(e_1, e_1) = 0, && (L^1_V g)(e_1, e_2) = -\lambda_2, \\ &(L^1_V g)(e_1, e_3) = -(b_2 + b_3)\lambda_2 - \lambda_3, && (L^1_V g)(e_2, e_2) = 2\lambda_1, \\ &(L^1_V g)(e_2, e_3) = \lambda_1(b_3 - b_1), && (L^1_V g)(e_3, e_3) = 0. \end{aligned} \tag{3.37}$$

If (G_4, g, J, V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases} -[1 + (\beta - 2\eta)(b_3 - b_1)] + \lambda = 0, \\ \lambda_2 = 0, \\ -(b_2 + b_3)\lambda_2 - \lambda_3 = 0, \\ \lambda_1 - [1 + (\beta - 2\eta)(b_2 + b_3)] + \lambda = 0, \\ \lambda_1(b_3 - b_1) + (\alpha + b_3 - b_1 - \beta) = 0, \\ \lambda = 0. \end{cases} \tag{3.38}$$

Solve (3.38), we have

Theorem 3.9. *(G_4, g, J, V) is not an affine Ricci soliton associated to the connection ∇^1.*

By (3.5) in [19], we have for (G_5, g, J, ∇^0), $\widetilde{\rho}^0(e_i, e_j) = 0$, for $1 \le i, j \le 3$. By Lemma 3.3 in [19] and (3.12), we have for (G_5, g, J, ∇^0, V)

$$\begin{aligned} &(L^0_V g)(e_1, e_1) = 0, && (L^0_V g)(e_1, e_2) = 0, \\ &(L^0_V g)(e_1, e_3) = \frac{\beta - \gamma}{2}\lambda_2, && (L^0_V g)(e_2, e_2) = 0, \\ &(L^0_V g)(e_2, e_3) = -\frac{\beta - \gamma}{2}\lambda_1, && (L^0_V g)(e_3, e_3) = 0. \end{aligned} \tag{3.39}$$

If (G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases} \lambda = 0, \\ (\beta - \gamma)\lambda_2 = 0, \\ (\beta - \gamma)\lambda_1 = 0, \end{cases} \tag{3.40}$$

Solve (3.40), we have

Theorem 3.10. *(G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^0 if and only if one of the following cases occurs*

(i) $\lambda = \beta = \gamma = 0, \alpha + \delta \neq 0.$
(ii) $\lambda = 0, \beta \neq \gamma, \lambda_1 = \lambda_2 = 0, \alpha + \delta \neq 0, \alpha\gamma + \beta\delta = 0.$

By Lemma 3.7 in [19], we have for (G_5, g, J, ∇^1), $\widetilde{\rho}^1(e_i, e_j) = 0$, for $1 \le i, j \le 3$. By Lemma 3.6 in [19] and (3.12), we have for (G_5, g, J, ∇^1, V)

$$\begin{aligned} &(L^1_V g)(e_1, e_1) = 0, && (L^1_V g)(e_1, e_2) = 0, \\ &(L^1_V g)(e_1, e_3) = -\alpha\lambda_1 - \gamma\lambda_2, && (L^1_V g)(e_2, e_2) = 0, \\ &(L^1_V g)(e_2, e_3) = -\beta\lambda_1 - \delta\lambda_2, && (L^1_V g)(e_3, e_3) = 0. \end{aligned} \tag{3.41}$$

If (G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases} \lambda = 0, \\ \alpha\lambda_1 + \gamma\lambda_2 = 0, \\ \beta\lambda_1 + \delta\lambda_2 = 0. \end{cases} \tag{3.42}$$

Solve (3.42), we have

Theorem 3.11. *(G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^1 if and only if the following statements hold true*

(i) $\lambda = \lambda_1 = \lambda_2 = 0,$
(ii) $\lambda = 0, \lambda_1 \neq 0, \lambda_2 = 0, \alpha = \beta = 0, \delta \neq 0,$
(iii) $\lambda = 0, \lambda_1 = 0, \lambda_2 \neq 0, \delta = \gamma = 0, \alpha \neq 0.$

By (3.18) in [19], we have for (G_6, g, J, ∇^0)

$$\widetilde{\rho}^0(e_1, e_1) = \frac{1}{2}\beta(\beta - \gamma) - \alpha^2, \qquad \widetilde{\rho}^0(e_1, e_2) = 0, \tag{3.43}$$

$$\widetilde{\rho}^0(e_1,e_3)=0, \qquad \widetilde{\rho}^0(e_2,e_2)=\frac{1}{2}\beta(\beta-\gamma)-\alpha^2,$$
$$\widetilde{\rho}^0(e_2,e_3)=\frac{1}{2}[-\gamma\alpha+\frac{1}{2}\delta(\beta-\gamma)], \qquad \widetilde{\rho}^0(e_3,e_3)=0.$$

By Lemma 3.11 in [19] and (3.12), we have for (G_6,g,J,∇^0,V)

$$\begin{aligned}(L^0_Vg)(e_1,e_1)&=0, & (L^0_Vg)(e_1,e_2)&=\alpha\lambda_2,\\ (L^0_Vg)(e_1,e_3)&=\frac{\gamma-\beta}{2}\lambda_2, & (L^0_Vg)(e_2,e_2)&=-2\alpha\lambda_1,\\ (L^0_Vg)(e_2,e_3)&=\frac{\beta-\gamma}{2}\lambda_1, & (L^0_Vg)(e_3,e_3)&=0.\end{aligned} \tag{3.44}$$

If (G_6,g,J,V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases}\frac{1}{2}\beta(\beta-\gamma)-\alpha^2+\lambda=0,\\ \alpha\lambda_2=0,\\ (\gamma-\beta)\lambda_2=0,\\ -\alpha\lambda_1+\frac{1}{2}\beta(\beta-\gamma)-\alpha^2+\lambda=0,\\ \frac{\beta-\gamma}{2}\lambda_1-\gamma\alpha+\frac{1}{2}\delta(\beta-\gamma)=0,\\ \lambda=0.\end{cases} \tag{3.45}$$

Solve (3.45), we have

Theorem 3.12. *(G_6,g,J,V) is an affine Ricci soliton associated to the connection ∇^0 if and only if*

(i) $\lambda=\lambda_1=\lambda_2=\gamma=\delta=0, \alpha\neq 0, \alpha^2=\frac{1}{2}\beta^2$,
(ii) $\lambda=\lambda_1=\lambda_2=\alpha=\beta=\gamma=0, \delta\neq 0$,
(iii) $\lambda=\lambda_2=0, \lambda_1\neq 0, \alpha=\beta=\gamma=0, \delta\neq 0$,
(iv) $\lambda=\lambda_2=0, \lambda_1\neq 0, \alpha=\beta=0, \delta\neq 0, \gamma\neq 0, \lambda_1=-\delta$,
(v) $\lambda=\alpha=\beta=\gamma=0, \lambda_2\neq 0, \delta\neq 0$.

By (3.23) in [19], we have for (G_6,g,J,∇^1)

$$\begin{aligned}\widetilde{\rho}^1(e_1,e_1)&=-(\alpha^2+\beta\gamma), & \widetilde{\rho}^1(e_1,e_2)&=0,\\ \widetilde{\rho}^1(e_1,e_3)&=0, & \widetilde{\rho}^1(e_2,e_2)&=-\alpha^2,\\ \widetilde{\rho}^1(e_2,e_3)&=0, & \widetilde{\rho}^1(e_3,e_3)&=0.\end{aligned} \tag{3.46}$$

By Lemma 3.15 in [19] and (3.12), we have for (G_6,g,J,∇^1,V)

$$\begin{aligned}(L^1_Vg)(e_1,e_1)&=0, & (L^1_Vg)(e_1,e_2)&=\lambda_2\alpha,\\ (L^1_Vg)(e_1,e_3)&=-\delta\lambda_3, & (L^1_Vg)(e_2,e_2)&=-2\alpha\lambda_1,\\ (L^1_Vg)(e_2,e_3)&=-\gamma\lambda_1, & (L^1_Vg)(e_3,e_3)&=0.\end{aligned} \tag{3.47}$$

If (G_6,g,J,V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases}-(\alpha^2+\beta\gamma)+\lambda=0,\\ \lambda_2\alpha=0,\\ \delta\lambda_3=0,\\ -\alpha\lambda_1-\alpha^2+\lambda=0,\\ \gamma\lambda_1=0,\\ \lambda=0.\end{cases} \tag{3.48}$$

Solve (3.48), we have

Theorem 3.13. *(G_6,g,J,V) is an affine Ricci soliton associated to the connection ∇^1 if and only if the following statements hold true*

(i) $\lambda=\alpha=\beta=\lambda_1=\lambda_3=0, \delta\neq 0$,
(ii) $\lambda=\alpha=\beta=\gamma=\lambda_3=0, \delta\neq 0, \lambda_1\neq 0$.

By (3.34) in [19], we have for (G_7,g,J,∇^0)

$$\widetilde{\rho}^0(e_1,e_1)=-\left(\alpha^2+\frac{\beta\gamma}{2}\right), \qquad \widetilde{\rho}^0(e_1,e_2)=0, \tag{3.49}$$

$$\widetilde{\rho}^0(e_1,e_3)=-\frac{1}{2}\left(\gamma\alpha+\frac{\delta\gamma}{2}\right),\quad \widetilde{\rho}^0(e_2,e_2)=-(\alpha^2+\frac{\beta\gamma}{2}),$$
$$\widetilde{\rho}^0(e_2,e_3)=\frac{1}{2}\left(\alpha^2+\frac{\beta\gamma}{2}\right),\quad \widetilde{\rho}^0(e_3,e_3)=0.$$

By Lemma 3.20 in [19] and (3.12), we have for (G_7,g,J,∇^0,V)

$$\begin{aligned}&(L^0_Vg)(e_1,e_1)=-2\alpha\lambda_2, && (L^0_Vg)(e_1,e_2)=\alpha\lambda_1-\beta\lambda_2,\\ &(L^0_Vg)(e_1,e_3)=(\beta-\frac{\gamma}{2})\lambda_2, && (L^0_Vg)(e_2,e_2)=2\beta\lambda_1,\\ &(L^0_Vg)(e_2,e_3)=(\frac{\gamma}{2}-\beta)\lambda_1, && (L^0_Vg)(e_3,e_3)=0.\end{aligned}\tag{3.50}$$

If (G_7,g,J,V) is an affine Ricci soliton associated to the connection ∇^0, then by (3.13), we have

$$\begin{cases}-\alpha\lambda_2-\left(\alpha^2+\frac{\beta\gamma}{2}\right)+\lambda=0,\\ \alpha\lambda_1-\beta\lambda_2=0,\\ \left(\beta-\frac{\gamma}{2}\right)\lambda_2-\left(\gamma\alpha+\frac{\delta\gamma}{2}\right)=0,\\ \beta\lambda_1-\left(\alpha^2+\frac{\beta\gamma}{2}\right)+\lambda=0,\\ \left(\frac{\gamma}{2}-\beta\right)\lambda_1+\alpha^2+\frac{\beta\gamma}{2}=0,\\ \lambda=0.\end{cases}\tag{3.51}$$

Solve (3.51), we have

Theorem 3.14. *(G_7,g,J,V) is an affine Ricci soliton associated to the connection ∇^0 if and only if the following statements hold true*

(i) $\lambda=\alpha=\beta=\gamma=0$, $\delta\neq 0$,
(ii) $\lambda=\alpha=\beta=0$, $\gamma\neq 0$, $\lambda_1=0$, $\lambda_2=-\delta$, $\delta\neq 0$,
(iii) $\lambda=\alpha=\gamma=\lambda_1=\lambda_2=0$, $\beta\neq 0$.

By (3.42) in [19], we have for (G_7,g,J,∇^1)

$$\begin{aligned}&\widetilde{\rho}^1(e_1,e_1)=-\alpha^2, && \widetilde{\rho}^1(e_1,e_2)=\frac{1}{2}(\beta\delta-\alpha\beta),\\ &\widetilde{\rho}^1(e_1,e_3)=\beta(\alpha+\delta), && \widetilde{\rho}^1(e_2,e_2)=-(\alpha^2+\beta^2+\beta\gamma),\\ &\widetilde{\rho}^1(e_2,e_3)=\frac{1}{2}(\beta\gamma+\alpha\delta+2\delta^2), && \widetilde{\rho}^1(e_3,e_3)=0.\end{aligned}\tag{3.52}$$

By Lemma 3.24 in [19] and (3.12), we have for (G_7,g,J,∇^1,V)

$$\begin{aligned}&(L^1_Vg)(e_1,e_1)=-2\alpha\lambda_2, && (L^1_Vg)(e_1,e_2)=\alpha\lambda_1-\beta\lambda_2,\\ &(L^1_Vg)(e_1,e_3)=-\alpha\lambda_1-\gamma\lambda_2-\beta\lambda_3, && (L^1_Vg)(e_2,e_2)=2\beta\lambda_1,\\ &(L^1_Vg)(e_2,e_3)=-\beta\lambda_1-\delta\lambda_2-\delta\lambda_3, && (L^1_Vg)(e_3,e_3)=0.\end{aligned}\tag{3.53}$$

If (G_7,g,J,V) is an affine Ricci soliton associated to the connection ∇^1, then by (3.14), we have

$$\begin{cases}-\alpha\lambda_2-\alpha^2+\lambda=0,\\ \alpha\lambda_1-\beta\lambda_2+\beta\delta-\alpha\beta=0,\\ -\alpha\lambda_1-\gamma\lambda_2-\beta\lambda_3+2\beta(\alpha+\delta)=0,\\ \beta\lambda_1-(\alpha^2+\beta^2+\beta\gamma)+\lambda=0,\\ -\beta\lambda_1-\delta\lambda_2-\delta\lambda_3+\beta\gamma+\alpha\delta+2\delta^2=0,\\ \lambda=0.\end{cases}\tag{3.54}$$

Solve (3.54), we have

Theorem 3.15. *(G_7,g,J,V) is an affine Ricci soliton associated to the connection ∇^1 if and only if*

(i) $\lambda=\alpha=\beta=\gamma=0$, $\lambda_2+\lambda_3-2\delta=0$, $\delta\neq 0$,
(ii) $\lambda=\alpha=\beta=0$, $\gamma\neq 0$, $\lambda_2=0$, $\lambda_3=2\delta$, $\delta\neq 0$,
(iii) $\lambda=\alpha=0$, $\delta\neq 0$, $\beta\neq 0$, $\lambda_1=\beta+\gamma$, $\lambda_2=\delta$, $\lambda_3=\frac{-\gamma\delta+2\beta\delta}{\beta}$, $\gamma=\frac{\beta(\beta^2+\delta^2)}{\delta^2}$.

4. AFFINE RICCI SOLITONS ASSOCIATED TO PERTURBED CANONICAL CONNECTIONS AND PERTURBED KOBAYASHI-NOMIZU CONNECTIONS ON THREE-DIMENSIONAL LORENTZIAN LIE GROUPS

We note that in our classifications in Section 2 always $\lambda = 0$. In order to get the affine Ricci soliton with non zero λ, we introduce perturbed canonical connections and perturbed Kobayashi-Nomizu connections in the following. Let e_3^* be the dual base of e_3. We define on $G_{i=1,...,7}$

$$\nabla^2_X Y = \nabla^0_X Y + \overline{\lambda} e_3^*(X) e_3^*(Y) e_3, \tag{4.1}$$

$$\nabla^3_X Y = \nabla^1_X Y + \overline{\lambda} e_3^*(X) e_3^*(Y) e_3, \tag{4.2}$$

where $\overline{\lambda}$ is a non zero real number. Then

$$\nabla^2_{e_3} e_3 = \overline{\lambda} e_3, \quad \nabla^2_{e_i} e_j = \nabla^0_{e_i} e_j; \tag{4.3}$$

$$\nabla^3_{e_3} e_3 = \overline{\lambda} e_3, \quad \nabla^3_{e_i} e_j = \nabla^1_{e_i} e_j. \tag{4.4}$$

where i or j does not equal 3. We let

$$(L^j_V g)(Y, Z) := g(\nabla^j_Y V, Z) + g(Y, \nabla^j_Z V), \tag{4.5}$$

for $j = 2, 3$ and vector fields V, Y, Z. Then we have for $G_{i=1,\cdots,7}$

$$(L^2_V g)(e_3, e_3) = -2\overline{\lambda}\lambda_3, \quad (L^2_V g)(e_j, e_k) = (L^0_V g)(e_j, e_k), \tag{4.6}$$

$$(L^3_V g)(e_3, e_3) = -2\overline{\lambda}\lambda_3, \quad (L^3_V g)(e_j, e_k) = (L^1_V g)(e_j, e_k), \tag{4.7}$$

where j or k does not equal 3.

Definition 4.1. *(G_i, g, J) is called the affine Ricci soliton associated to the connection ∇^2 if it satisfies*

$$(L^2_V g)(Y, Z) + 2\widetilde{\rho}^2(Y, Z) + 2\lambda g(Y, Z) = 0. \tag{4.8}$$

(G_i, g, J) is called the affine Ricci soliton associated to the connection ∇^3 if it satisfies

$$(L^3_V g)(Y, Z) + 2\widetilde{\rho}^3(Y, Z) + 2\lambda g(Y, Z) = 0. \tag{4.9}$$

For (G_1, ∇^2), similar to (3.15), we have

$$\widetilde{\rho}^2(e_2, e_3) = \frac{\alpha^2 + \overline{\lambda}\alpha}{2}, \quad \widetilde{\rho}^2(e_j, e_k) = \widetilde{\rho}^0(e_j, e_k), \tag{4.10}$$

for the pair $(j, k) \neq (2, 3)$. If (G_1, g, J, V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} 2\lambda_2\alpha - 2\alpha^2 - \beta^2 + 2\lambda = 0, \\ \lambda_1\alpha = 0, \\ -\beta\lambda_2 + \alpha\beta = 0, \\ -2\alpha^2 - \beta^2 + 2\lambda = 0, \\ \frac{\beta}{2}\lambda_1 + \alpha^2 + \overline{\lambda}\alpha = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.11}$$

Solve (4.11), we have

Theorem 4.2. *(G_1, g, J, V) is an affine Ricci soliton associated to the connection ∇^2 if and only if $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -\overline{\lambda}$, $\alpha = -\overline{\lambda}$, $\beta = 0$, $\lambda = \overline{\lambda}^2$.*

For (G_1, ∇^3), similar to (3.18), we have

$$\widetilde{\rho}^3(e_2, e_3) = \frac{\alpha^2 + \overline{\lambda}\alpha}{2}, \quad \widetilde{\rho}^3(e_j, e_k) = \widetilde{\rho}^1(e_j, e_k), \tag{4.12}$$

for the pair $(j, k) \neq (2, 3)$. If (G_1, g, J, V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} \lambda_2\alpha - \alpha^2 - \beta^2 + \lambda = 0, \\ -\lambda_1\alpha + 2\alpha\beta = 0, \\ \lambda_1\alpha - \beta\lambda_2 - \alpha\beta = 0, \\ -\alpha^2 - \beta^2 + \lambda = 0, \\ \beta\lambda_1 - \alpha\lambda_2 - \alpha\lambda_3 + \alpha^2 + \overline{\lambda}\alpha = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.13}$$

Solve (4.13), we have

Theorem 4.3. *(G_1, g, J, V) is not an affine Ricci soliton associated to the connection ∇^3.*

Proof. By the first and second and fourth equations in (4.13) and $\alpha \neq 0$, we get $\lambda_2 = 0$, $\lambda_1 = 2\beta$, $\lambda = \alpha^2 + \beta^2$, By the third equation in (4.13), we get $\lambda_1 = \lambda_2 = \beta = 0$, $\lambda = \alpha^2$. By the fifth equation in (4.13), we get $\lambda_3 = \alpha + \bar{\lambda}$. By the sixth equation in (4.13), we get $\alpha^2 + \bar{\lambda}\alpha + \bar{\lambda}^2 = 0$. Then $\bar{\lambda} = \alpha = 0$, this is a contradiction. □

For (G_2, ∇^2), similar to (3.21), we have

$$\widetilde{\rho}^2(e_1, e_3) = \frac{-\gamma\bar{\lambda}}{2}, \quad \widetilde{\rho}^2(e_j, e_k) = \widetilde{\rho}^0(e_j, e_k), \tag{4.14}$$

for the pair $(j, k) \neq (1, 3)$. If (G_2, g, J, V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} -\left(\gamma^2 + \frac{\alpha\beta}{2}\right) + \lambda = 0, \\ \lambda_2\gamma = 0, \\ \alpha\lambda_2 + 2\gamma\bar{\lambda} = 0, \\ -\gamma\lambda_1 - \left(\gamma^2 + \frac{\alpha\beta}{2}\right) + \lambda = 0, \\ \frac{\alpha}{2}\lambda_1 + 2\left(\frac{\beta\gamma}{2} - \frac{\alpha\gamma}{4}\right) = 0, \\ \bar{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.15}$$

Solve (4.15), we have

Theorem 4.4. *(G_2, g, J, V) is not an affine Ricci soliton associated to the connection ∇^2.*

For (G_2, ∇^3), similar to (3.24), we have

$$\widetilde{\rho}^3(e_1, e_3) = \frac{-\gamma\bar{\lambda}}{2}, \quad \widetilde{\rho}^3(e_j, e_k) = \widetilde{\rho}^1(e_j, e_k), \tag{4.16}$$

for the pair $(j, k) \neq (1, 3)$. If (G_2, g, J, V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} -\beta^2 - \gamma^2 + \lambda = 0, \\ \lambda_2\gamma = 0, \\ -\alpha\lambda_2 + \gamma\lambda_3 - \gamma\bar{\lambda} = 0, \\ -\gamma\lambda_1 - \left(\gamma^2 + \alpha\beta\right) + \lambda = 0, \\ \lambda_1\beta - \alpha\gamma = 0, \\ \bar{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.17}$$

Solve (4.17), we have

Theorem 4.5. *(G_2, g, J, V) is not an affine Ricci soliton associated to the connection ∇^3.*

For (G_3, ∇^2), we have $\widetilde{\rho}^2(e_j, e_k) = \widetilde{\rho}^0(e_j, e_k)$, for any pairs (j, k). If (G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} -\gamma a_3 + \lambda = 0, \\ \lambda_2 a_3 = 0, \\ \lambda_1 a_3 = 0, \\ \bar{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.18}$$

Solve (4.18), we have

Theorem 4.6. *(G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^2 if and only if the following statements hold true*

(i) $a_3 \neq 0$, $\lambda_1 = \lambda_2 = 0$, $\lambda = \gamma a_3$, $\lambda_3 = -\frac{\gamma a_3}{\bar{\lambda}}$,
(ii) $a_3 = \lambda = \lambda_3 = 0$.

For (G_3, ∇^3), we have $\widetilde{\rho}^3(e_j, e_k) = \widetilde{\rho}^1(e_j, e_k)$, for any pairs (j, k). If (G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} \gamma(a_1 - a_3) + \lambda = 0, \\ (a_2 + a_3)\lambda_2 = 0, \\ -\gamma(a_2 + a_3) + \lambda = 0, \\ \lambda_1(a_3 - a_1) = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.19}$$

Solve (4.19), we have

Theorem 4.7. *(G_3, g, J, V) is an affine Ricci soliton associated to the connection ∇^3 if and only if one of the following cases occurs*

(i) $\gamma = \lambda = \lambda_3 = 0, \alpha\lambda_2 = 0, \beta\lambda_1 = 0,$
(ii) $\gamma \neq 0, \alpha = \beta = \lambda = \lambda_3 = 0,$
(iii) $\gamma \neq 0, \alpha = \beta \neq 0, \lambda_1 = \lambda_2 = 0, \lambda = \alpha\gamma, \lambda_3 = -\frac{\alpha\gamma}{\overline{\lambda}}.$

For (G_4, ∇^2), we have

$$\widetilde{\rho}^2(e_1, e_3) = \frac{\overline{\lambda}}{2}, \quad \widetilde{\rho}^2(e_j, e_k) = \widetilde{\rho}^0(e_j, e_k), \tag{4.20}$$

for the pair $(j, k) \neq (1, 3)$. If (G_4, g, J, V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} (2\eta - \beta)b_3 - 1 + \lambda = 0, \\ \lambda_2 = 0, \\ -b_3\lambda_2 + \overline{\lambda} = 0, \\ \lambda_1 + (2\eta - \beta)b_3 - 1 + \lambda = 0, \\ \lambda_1 b_3 + b_3 - \beta = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.21}$$

Solve (4.21), we have

Theorem 4.8. *(G_4, g, J, V) is not an affine Ricci soliton associated to the connection ∇^2.*

For (G_4, ∇^3), we have

$$\widetilde{\rho}^3(e_1, e_3) = \frac{\overline{\lambda}}{2}, \quad \widetilde{\rho}^3(e_j, e_k) = \widetilde{\rho}^1(e_j, e_k), \tag{4.22}$$

for the pair $(j, k) \neq (1, 3)$. If (G_4, g, J, V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} -[1 + (\beta - 2\eta)(b_3 - b_1)] + \lambda = 0, \\ \lambda_2 = 0, \\ -(b_2 + b_3)\lambda_2 - \lambda_3 + \overline{\lambda} = 0, \\ \lambda_1 - [1 + (\beta - 2\eta)(b_2 + b_3)] + \lambda = 0, \\ \lambda_1(b_3 - b_1) + (\alpha + b_3 - b_1 - \beta) = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.23}$$

Solve (4.23), we have

Theorem 4.9. *(G_4, g, J, V) is not an affine Ricci soliton associated to the connection ∇^3.*

For (G_5, g, J, ∇^2), $\widetilde{\rho}^2(e_i, e_j) = 0$, for $1 \leq i, j \leq 3$. If (G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} \lambda = 0, \\ (\beta - \gamma)\lambda_2 = 0, \\ (\beta - \gamma)\lambda_1 = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.24}$$

Solve (4.24), we have

Theorem 4.10. *(G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^2 if and only if the following statements hold true*

(i) $\gamma \neq \beta, \lambda = \lambda_1 = \lambda_2 = \lambda_3 = 0, \alpha + \delta \neq 0, \alpha\gamma + \beta\delta = 0,$
(ii) $\lambda = \beta = \gamma = 0, \alpha + \delta \neq 0, \lambda_3 = 0.$

For (G_5, g, J, ∇^3), $\widetilde{\rho}^3(e_i, e_j) = 0$, for $1 \le i, j \le 3$. If (G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} \lambda = 0, \\ \alpha\lambda_1 + \gamma\lambda_2 = 0, \\ \beta\lambda_1 + \delta\lambda_2 = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.25}$$

Solve (4.25), we have

Theorem 4.11. *(G_5, g, J, V) is an affine Ricci soliton associated to the connection ∇^3 if and only if*

(i) $\lambda = \lambda_1 = \lambda_2 = \lambda_3 = 0$,
(ii) $\lambda = \lambda_2 = \lambda_3 = \alpha = \beta = 0$, $\lambda_1 \neq 0$, $\delta \neq 0$,
(iii) $\lambda = 0$, $\lambda_1 = \lambda_3 = 0$, $\lambda_2 \neq 0$, $\delta = \gamma = 0$, $\alpha \neq 0$.

For (G_6, ∇^2), we have

$$\widetilde{\rho}^2(e_1, e_3) = \frac{\delta\overline{\lambda}}{2}, \qquad \widetilde{\rho}^2(e_j, e_k) = \widetilde{\rho}^0(e_j, e_k), \tag{4.26}$$

for the pair $(j, k) \neq (1, 3)$. If (G_6, g, J, V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} \frac{1}{2}\beta(\beta - \gamma) - \alpha^2 + \lambda = 0, \\ \alpha\lambda_2 = 0, \\ (\gamma - \beta)\lambda_2 + 2\delta\overline{\lambda} = 0, \\ -\alpha\lambda_1 + \frac{1}{2}\beta(\beta - \gamma) - \alpha^2 + \lambda = 0, \\ \frac{\beta - \gamma}{2}\lambda_1 - \gamma\alpha + \frac{1}{2}\delta(\beta - \gamma) = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.27}$$

Solve (4.27), we have

Theorem 4.12. *(G_6, g, J, V) is an affine Ricci soliton associated to the connection ∇^2 if and only if the following statements hold true*

(i) $\alpha = \beta = 0$, $\delta \neq 0$, $\gamma \neq 0$, $\lambda = \lambda_3 = 0$, $\lambda_1 = -\delta$, $\lambda_2 = -\frac{2\delta\overline{\lambda}}{\gamma}$,
(ii) $\alpha \neq 0$, $\lambda_1 = \lambda_2 = \gamma = \delta = 0$, $\lambda = \alpha^2 - \frac{1}{2}\beta^2$, $\lambda_3 = -\frac{\lambda}{\overline{\lambda}}$.

For (G_6, ∇^3), we have

$$\widetilde{\rho}^3(e_1, e_3) = \frac{\delta\overline{\lambda}}{2}, \qquad \widetilde{\rho}^3(e_j, e_k) = \widetilde{\rho}^1(e_j, e_k), \tag{4.28}$$

for the pair $(j, k) \neq (1, 3)$. If (G_6, g, J, V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} -(\alpha^2 + \beta\gamma) + \lambda = 0, \\ \lambda_2\alpha = 0, \\ -\delta\lambda_3 + \delta\overline{\lambda} = 0, \\ -\alpha\lambda_1 - \alpha^2 + \lambda = 0, \\ \gamma\lambda_1 = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.29}$$

Solve (4.29), we have

Theorem 4.13. *(G_6, g, J, V) is an affine Ricci soliton associated to the connection ∇^3 if and only if* $\alpha \neq 0$, $\lambda_1 = \lambda_2 = \gamma = \delta = 0$, $\lambda = \alpha^2$, $\lambda_3 = -\frac{\alpha^2}{\overline{\lambda}}$.

For (G_7, ∇^2), we have

$$\widetilde{\rho}^2(e_1, e_3) = \frac{1}{2}(\beta\overline{\lambda} - \alpha\gamma - \frac{\delta\gamma}{2}), \qquad \widetilde{\rho}^2(e_2, e_3) = \frac{1}{2}(\delta\overline{\lambda} + \alpha^2 + \frac{\beta\gamma}{2}), \qquad \widetilde{\rho}^2(e_j, e_k) = \widetilde{\rho}^0(e_j, e_k), \tag{4.30}$$

for the pair $(j,k) \neq (1,3),(2,3)$. If (G_7,g,J,V) is an affine Ricci soliton associated to the connection ∇^2, then by (4.8), we have

$$\begin{cases} -\alpha\lambda_2 - \left(\alpha^2 + \frac{\beta\gamma}{2}\right) + \lambda = 0, \\ \alpha\lambda_1 - \beta\lambda_2 = 0, \\ \left(\beta - \frac{\gamma}{2}\right)\lambda_2 + \beta\overline{\lambda} - \left(\gamma\alpha + \frac{\delta\gamma}{2}\right) = 0, \\ \beta\lambda_1 - \left(\alpha^2 + \frac{\beta\gamma}{2}\right) + \lambda = 0, \\ \left(\frac{\gamma}{2} - \beta\right)\lambda_1 + \delta\overline{\lambda} + \alpha^2 + \frac{\beta\gamma}{2} = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.31}$$

Solve (4.31), we have

Theorem 4.14. *(G_7,g,J,V) is an affine Ricci soliton associated to the connection ∇^2 if and only if one of the following cases occurs*

(i) $\alpha = \beta = 0, \gamma \neq 0, \delta \neq 0, \lambda = 0, \lambda_1 = -\frac{2\delta\overline{\lambda}}{\gamma}, \lambda_2 = -\delta, \lambda_3 = 0,$
(ii) $\alpha \neq 0, \lambda_1 = \lambda_2 = \beta = \gamma = 0, \lambda = \alpha^2, \delta \neq 0, \overline{\lambda} = -\frac{\alpha^2}{\delta}, \lambda_3 = \delta.$

For (G_7, ∇^3), we have

$$\widetilde{\rho}^3(e_1,e_3) = \alpha\beta + \beta\delta + \frac{\beta\overline{\lambda}}{2}, \qquad \widetilde{\rho}^3(e_2,e_3) = \frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2 + \delta\overline{\lambda}), \qquad \widetilde{\rho}^3(e_j,e_k) = \widetilde{\rho}^1(e_j,e_k), \tag{4.32}$$

for the pair $(j,k) \neq (1,3),(2,3)$. If (G_7,g,J,V) is an affine Ricci soliton associated to the connection ∇^3, then by (4.9), we have

$$\begin{cases} -\alpha\lambda_2 - \alpha^2 + \lambda = 0, \\ \alpha\lambda_1 - \beta\lambda_2 + \beta\delta - \alpha\beta = 0, \\ -\alpha\lambda_1 - \gamma\lambda_2 - \beta\lambda_3 + 2\beta(\alpha + \delta + \frac{\overline{\lambda}}{2}) = 0, \\ \beta\lambda_1 - (\alpha^2 + \beta^2 + \beta\gamma) + \lambda = 0, \\ -\beta\lambda_1 - \delta\lambda_2 - \delta\lambda_3 + \beta\gamma + \alpha\delta + 2\delta^2 + \delta\overline{\lambda} = 0, \\ \overline{\lambda}\lambda_3 + \lambda = 0. \end{cases} \tag{4.33}$$

Solve (4.33), we have

Theorem 4.15. *(G_7,g,J,V) is an affine Ricci soliton associated to the connection ∇^3 if and only if the following statements hold true*

(i) $\lambda = \alpha = \beta = \gamma = \lambda_3 = 0, \delta \neq 0, \lambda_2 = 2\delta + \overline{\lambda},$
(ii) $\alpha = \beta = \lambda = \lambda_2 = \lambda_3 = 0, \gamma \neq 0, \delta \neq 0, \overline{\lambda} = -2\delta,$
(iii) $\alpha = \lambda = \lambda_3 = 0, \beta \neq 0, \delta \neq 0, \lambda_1 = \beta + \gamma, \lambda_2 = \delta, \overline{\lambda} = \frac{\gamma\delta - 2\beta\delta}{\beta}, \gamma = \frac{\beta^3 + \beta\delta^2}{\delta^2},$
(iv) $\alpha \neq 0, \beta = \gamma = \delta = \lambda_1 = \lambda_2 = 0, \lambda = \alpha^2, \lambda_3 = -\frac{\alpha^2}{\overline{\lambda}},$
(v) $\alpha \neq 0, \beta = \gamma = \lambda_1 = \lambda_2 = 0, \lambda = \alpha^2, \delta \neq 0, \lambda_3 = \alpha + 2\delta + \overline{\lambda}, \overline{\lambda}^2 + (\alpha + 2\delta)\overline{\lambda} + \alpha^2 = 0.$

Proof. We know that $\alpha\gamma = 0$ and $\alpha + \delta \neq 0$.

Case i) $\alpha = 0$, then $\delta \neq 0$. By (4.33), we have $\lambda = \lambda_3 = 0$ and

$$\begin{cases} \beta(\lambda_2 - \delta) = 0, \\ -\gamma\lambda_2 + 2\beta\delta + \beta\overline{\lambda} = 0, \\ \beta\lambda_1 - (\beta^2 + \beta\gamma) = 0, \\ -\beta\lambda_1 - \delta\lambda_2 + \beta\gamma + 2\delta^2 + \delta\overline{\lambda} = 0. \end{cases} \tag{4.34}$$

Case i)-a) $\beta = 0$, then by (4.34), we have $\gamma\lambda_2 = 0$ and $\lambda_2 = 2\delta + \overline{\lambda}$.
Case i)-a)-1) $\gamma = 0$, we get (i).
Case i)-a)-2) $\gamma \neq 0$, we get $\lambda_2 = 0$ and $\overline{\lambda} = -2\delta$. So we have (ii).
Case i)-b) $\beta \neq 0$, then by (4.34), we have $\lambda_1 = \beta + \gamma, \lambda_2 = \delta, \overline{\lambda} = \frac{\gamma\delta - 2\beta\delta}{\beta}$. By the fourth equation in (4.34), we get $\gamma = \frac{\beta^3 + \beta\delta^2}{\delta^2}$ and this is (iii).
Case ii) $\alpha \neq 0$, so $\gamma = 0$.
Case ii)-a) $\beta = 0$, by (4.33), we get $\lambda_1 = \lambda_2 = 0, \lambda = \alpha^2, \delta\lambda_3 = \alpha\delta + 2\delta^2 + \delta\overline{\lambda}, \lambda_3 = -\frac{\alpha^2}{\overline{\lambda}}$.
Case ii)-a)-1) $\delta = 0$, we get (iv).
Case ii)-a)-2) $\delta \neq 0$, we get (v).

Case ii)-b) $\beta \neq 0$, we get $\alpha\lambda_2 + \beta\lambda_1 - \beta^2 = 0$ and $\alpha\lambda_1 - \beta\lambda_2 + \beta(\delta - \alpha) = 0$. So get

$$\begin{cases} \lambda_1 = \beta - \frac{\alpha\beta\delta}{\alpha^2+\beta^2}, \\ \lambda_2 = \frac{\beta^2\delta}{\alpha^2+\beta^2}, \\ \lambda = \frac{\alpha\beta^2\delta}{\alpha^2+\beta^2} + \alpha^2, \\ \lambda_3 = \overline{\lambda} + \alpha + 2\delta + \frac{\alpha^2\delta}{\alpha^2+\beta^2}. \end{cases} \tag{4.35}$$

Using (4.35) and the fifth equation in (4.33), we get $\beta^2(\alpha^2 - \alpha\delta + \delta^2 + \beta^2) + \alpha^2\delta^2 = 0$, so we get $\beta = 0$ and this is a contradiction. So we have no solutions in this case. □

CONFLICTS OF INTEREST

The author declares no conflicts of interest.

FUNDING

This research was funded by National Natural Science Foundation of China: No. 11771070.

ACKNOWLEDGMENTS

The author was supported in part by NSFC No. 11771070. The author is deeply grateful to the referees for their valuable comments and helpful suggestions.

REFERENCES

[1] Z.M. Balogh, J.T. Tyson, E. Vecchi, Intrinsic curvature of curves and surfaces and a Gauss-Bonnet theorem in the Heisenberg group, Math. Z. 287 (2017), 1–38.
[2] W. Batat, K. Onda, Algebraic Ricci solitons of three-dimensional Lorentzian Lie groups, J. Geom. Phys. 114 (2017), 138–152.
[3] G. Calvaruso, Three-dimensional homogeneous generalized Ricci solitons, Mediterr. J. Math. 14 (2017), 216.
[4] G. Calvaruso, Homogeneous structures on three-dimensional Lorentzian manifolds, J. Geom. Phys. 57 (2007), 1279–1291.
[5] L.A. Cordero, P.E. Parker, Left-invariant Lorentzian metrics on 3-dimensional Lie groups, Rend. Mat. Appl. 17 (1997), 129–155.
[6] M. Crasmareanu, A new approach to gradient Ricci solitons and generalizations, Filomat 32 (2018), 3337–3346.
[7] F. Etayo, R. Santamaría, Distinguished connections on $(J^2 = \pm 1)$-metric manifolds, Arch. Math. (Brno) 52 (2016), 159–203.
[8] N. Halammanavar, K. Devasandra, Kenmotsu manifolds admitting Schouten-van Kampen connection, Facta Univ. Ser. Math. Inform. 34 (2019), 23–34.
[9] R.S. Hamilton, The Ricci flow on surfaces, in: J.A. Isenberg, (Ed.), Mathematics and general relativity, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, Santa Cruz, CA, 1988, pp. 237–262.
[10] Y. Han, A. De, P. Zhao, On a semi-quasi-Einstein manifold, J. Geom. Phys. 155 (2020), 103739.
[11] S. Hui, R. Prasad, D. Chakraborty, Ricci solitons on Kenmotsu manifolds with respect to quarter symmetric non-metric ϕ-connection, Ganita 67 (2017), 195–204.
[12] S.Y. Perktas, A. Yildiz, On quasi-Sasakian 3-manifolds with respect to the Schouten-van Kampen connection, Int. Elec. J. Geom. 13 (2020), 62–74.
[13] Q. Qu, Y. Wang, Multiply warped products with a quarter-symmetric connection, J. Math. Anal. Appl. 431 (2015), 955–987.
[14] A.N. Siddiqui, B.Y. Chen, O. Bahadir, Statistical solitons and inequalities for statistical warped product submanifolds, Mathematics 7 (2019), 797.
[15] S. Sular, C. Özgür, Warped products with a semi-symmetric metric connection, Taiwanese J. Math. 15 (2011), 1701–1719.
[16] S. Sular, C. Özgür, Warped products with a semi-symmetric non-metric connection, Arab. J. Sci. Eng. 36 (2011), 461–473.
[17] Y. Wang, Multiply warped products with a semisymmetric metric connection, Abstr. Appl. Anal. 2014 (2014), 742371.
[18] Y. Wang, Curvature of multiply warped products with an affine connection, Bull. Korean Math. Soc. 50 (2013), 1567–1586.
[19] Y. Wang, Canonical connections and algebraic Ricci solitons of three-dimensional Lorentzian Lie groups, arxiv:2001.11656.2020.
[20] Y. Wang, S. Wei, Gauss-Bonnet theorems in the affine group and the group of rigid motions of the Minkowski plane, Sci. China Math. 2020.
[21] T. Wu, S. Wei, Y. Wang, Gauss-Bonnet theorems and the Lorentzian Heisenberg group, 2020, Available from: https://www.researchgate.net/publication/345360459_GAUSS-BONNET_THEOREMS_AND_THE_LORENTZIAN_HEISENBERG_GROUP.

Journal of Nonlinear Mathematical Physics
Vol. 28(3); September (2021), pp. 292–302
DOI: https://doi.org/10.2991/jnmp.k.210330.001; ISSN 1402-9251; eISSN 1776-0852
https://www.atlantis-press.com/journals/jnmp

Research Article

The Orthogonal and Symplectic Schur Functions, Vertex Operators and Integrable Hierarchies

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ARTICLE INFO

Article History

Received 18 March 2018
Accepted 10 March 2021

Keywords

Boson-Fermion correspondence
orthogonal and symplectic schur functions
integrable hierarchy
universal character

ABSTRACT

In this paper, we first construct an integrable system whose solutions include the orthogonal Schur functions and the symplectic Schur functions. We find that the orthogonal Schur functions and the symplectic Schur functions can be obtained by one kind of Boson-Fermion correspondence which is slightly different from the classical one. Then, we construct a universal character which satisfies the bilinear equation of a new infinite-dimensional integrable orthogonal UC hierarchy.

1. INTRODUCTION

Boson-Fermion correspondence is well-known in mathematical physics [1,11]. Young diagrams and symmetric functions are of interest to many researchers and have many applications in mathematics including combinatorics and representation theory [3,10]. There are many relations between Boson-Fermion correspondence and symmetric functions.

The KP hierarchy [1] is one of the most important integrable hierarchies and it arises in many different fields of mathematics and physics such as enumerative algebraic geometry, topological field and string theory. Schur functions have close relations with the τ-functions of KP hierarchy. Schur functions give the characters of finite-dimensional irreducible representations of the general linear groups, see [3,10]. Schur functions can be realized from vertex operators as in Equation (6) of this paper, and these vertex operators can be used to construct Fermions which act on Bosonic Fock space, see [7,11]. By replacing nx_n by power sum, we find that the character of Young diagram in [11] is the same with the Schur function obtained from the Jacobi-Trudi formula, which tells us that the Schur functions are solutions of differential equations in the KP hierarchy, and the linear combinations of Schur functions with coefficients satisfying some relations (plücker relations) are also τ-functions of the KP hierarchy. In [12,13], the author generalized the KP hierarchy to the UC (universal character) hierarchy, whose τ-functions include universal characters [8].

The orthogonal and symplectic Schur functions are upgraded from Schur functions in the same setting [2]. Symplectic Schur functions are equal to orthogonal Schur functions with the conjugate Young diagrams. Like Schur functions, the symplectic and orthogonal Schur functions can also be realized from vertex operators as in Equation (19), and these vertex operators can also be used to construct Fermions. Then there certainly exists an integrable system. In this paper, we will construct this integrable system, and show that the symplectic and orthogonal Schur functions are its solutions.

This paper is arranged as follows. In Section 2, we will recall the definition of Schur function, its vertex operator realization, and the relations between Schur functions and KP hierarchy. In Section 3, we will recall the definitions of orthogonal and symplectic Schur functions, their respective vertex operator realization, then we will define an integrable system whose τ-function can be obtained from orthogonal and symplectic Schur function. In Section 4, we will construct a method to calculate orthogonal and symplectic Schur functions from a different kind of Boson-Fermion correspondence. In Section 5, we will construct the modified type of the integrable system which is constructed in Section 3. In Section 6, we will consider the universal character and the corresponding UC hierarchy.

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2. SCHUR FUNCTIONS, VERTEX OPERATOR AND THE KP HIERARCHY

Let $\mathbf{x} = (x_1, x_2, \cdots)$. The operators $h_n(\mathbf{x})$ are determined by the generating function:

$$e^{\xi(\mathbf{x},k)} := \sum_{n=0}^{\infty} h_n(\mathbf{x})k^n, \quad \text{where } \xi(\mathbf{x},k) = \sum_{n=1}^{\infty} x_n k^n \tag{1}$$

and set $h_n(\mathbf{x}) = 0$ for $n < 0$. Note that if we replace ix_i with the power sum $p_i = \sum_n x_n^i$, $h_n(\mathbf{x})$ is the complete homogeneous symmetric function [10]

$$\sum_{i_1 \le i_2 \le \cdots \le i_n} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{2}$$

For Young diagrams $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, the Schur function $S_\lambda = S_\lambda(\mathbf{x})$ is a polynomial in $\mathbb{C}[\mathbf{x}]$ defined by the Jacobi-Trudi formula [8]:

$$S_\lambda(\mathbf{x}) = \det\left(h_{\lambda_i - i + j}(\mathbf{x})\right)_{1 \le i,j \le l}. \tag{3}$$

Introduce the following vertex operators

$$V^+(k) = \sum_{n \in \mathbb{Z}} V_n^+ k^n = e^{\xi(\mathbf{x},k)} e^{-\xi(\tilde{\partial}_{\mathbf{x}}, z^{-1})}, \tag{4}$$

$$V^-(k) = \sum_{n \in \mathbb{Z}} V_n^- k^n = e^{-\xi(\mathbf{x},k)} e^{\xi(\tilde{\partial}_{\mathbf{x}}, k^{-1})}. \tag{5}$$

where $\tilde{\partial}_{\mathbf{x}} = (\partial_{x_1}, \frac{1}{2}\partial_{x_2}, \cdots, \frac{1}{n}\partial_{x_n}, \cdots)$. The operators V_i^+ are raising operators for the Schur functions

$$S_\lambda(\mathbf{x}) := V_{\lambda_1}^+ \cdots V_{\lambda_l}^+ \cdot 1 \tag{6}$$

where λ is a Young diagram $(\lambda_1, \lambda_2, \cdots, \lambda_l)$, and we denote $S_\lambda(\mathbf{x})$ by S_λ for short.

Introduce Fermions ψ_j^* and ψ_j for any $j \in \mathbb{Z} + \frac{1}{2}$ as operators satisfying the relations

$$\{\psi_j, \psi_k\} = 0, \ \{\psi_j^*, \psi_k^*\} = 0, \ \{\psi_j^*, \psi_k\} = \delta_{j+k,0} \tag{7}$$

where $\{A, B\} = AB + BA$. The generating functions of Fermions are

$$\psi(k) := \sum_{j \in \mathbb{Z}+1/2} \psi_j k^{-j-1/2}, \ \psi^*(k) := \sum_{j \in \mathbb{Z}+1/2} \psi_j^* k^{-j-1/2}.$$

The Fock representation space of Fermions is the space of Maya diagrams. A Maya diagram is made up of black and white stones lined up along the real line with the convention that all the stones are black far away to the right, whereas all the stones are white far away to the left. For example, the following is a Maya diagram

$$\begin{array}{ccccccccccc} \cdots & \circ & \circ & \bullet & \bullet & \circ & \bullet & \circ & \bullet & \bullet & \cdots \\ & -\frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \end{array} \tag{8}$$

By writing half integers $u_1, u_2, \cdots$ for the positions of the black stones, a Maya diagram is described as an increasing sequence of half integers

$$\mathbf{u} = \{u_n\}_{n \ge 1} \quad \text{with} \quad u_1 < u_2 < u_3 < \cdots.$$

For example, the Maya diagram in (8) is denoted by

$$-\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \frac{9}{2}, \cdots$$

Define the charge p of a Maya diagram as the number of white stones on the right half line minus the number of black stones on the left half line. For example, the charge of Maya diagram in (8) is zero.

Let $\mathcal{F}$ be the vector space based by the set of Maya diagrams, which is called Fermionic Fock space. The basis vector is written as $|\mathbf{u}\rangle$. In particular,

$$|0\rangle = |\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots\rangle.$$

The action of Fermions ψ_j and ψ_j^* for any $j \in \frac{1}{2} + \mathbb{Z}$ on Maya diagrams $|\mathbf{u}\rangle$ is determined by the formulas

$$\psi_j|\mathbf{u}\rangle = \begin{cases} (-1)^{i-1}|\cdots, u_{i-1}, u_{i+1}, \cdots\rangle & \text{if } u_i = -j \text{ for some } i, \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

$$\psi_j^*|\mathbf{u}\rangle = \begin{cases} (-1)^i|\cdots, u_i, j, u_{i+1}, \cdots\rangle & \text{if } u_i < j < u_{i+1} \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

There are three vector spaces which are isomorphic to each other [11]: the polynomial ring $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, x_2, \cdots]$ of infinitely many variables $\mathbf{x} = (x_1, x_2, \cdots)$ which is called the Bosonic Fock space, the charge zero part of the Fermionic Fock space $\mathcal{F}$, and the vector space Y spanned by Young diagrams. Therefore, the Maya diagram $|\mathbf{u}\rangle$ can be written as

$$|\mathbf{u}\rangle = |\lambda, n\rangle = |S_\lambda, n\rangle,$$

where n is the charge of $|\mathbf{u}\rangle$. In the special case of $n = 0$, we also write the Maya diagram $|\mathbf{u}\rangle$ as $|\lambda\rangle$.

Let $f(z, x) \in C[z, z^{-1}, x_1, x_2, \cdots]$. Define operators

$$e^K f(z, \mathbf{x}) := zf(z, \mathbf{x}), \quad k^{H_0} f(z, \mathbf{x}) := f(kz, \mathbf{x}). \tag{11}$$

Define the generating functions [5,11]

$$\tilde{V}(k) := \sum_{j\in\mathbb{Z}+\frac{1}{2}} \tilde{V}_j k^{-j-\frac{1}{2}} = V^+(k)e^K k^{H_0}, \tag{12}$$

$$\tilde{V}^*(k) := \sum_{j\in\mathbb{Z}+\frac{1}{2}} \tilde{V}_j^* k^{-j-\frac{1}{2}} = V^-(k)e^K k^{H_0}. \tag{13}$$

It can be checked that

$$\{\tilde{V}_i, \tilde{V}_j\} = 0, \ \{\tilde{V}_i^*, \tilde{V}_j^*\} = 0, \ \{\tilde{V}_i, \tilde{V}_j^*\} = \delta_{i+j,0}, \tag{14}$$

that is, the operators $\tilde{V}_i$, $\tilde{V}_j^*$ determine a representation of the algebra spanned by Fermions, see equations in (7).

Definition 2.1. *For an unknown function $\tau = \tau(\mathbf{x})$, the bilinear equation*

$$\sum_{j\in\mathbb{Z}+\frac{1}{2}} \tilde{V}_j^* \tau \otimes \tilde{V}_{-j}\tau = 0 \tag{15}$$

is called the KP hierarchy, see [5,11].

3. THE ORTHOGONAL SCHUR FUNCTION, THE SYMPLECTIC SCHUR FUNCTION, VERTEX OPERATORS AND AN INTEGRABLE HIERARCHY

For a Young diagram $\lambda = (\lambda_1, \cdots, \lambda_l)$, the orthogonal Schur function[6,9] is defined to be

$$S_\lambda^O := \det(h_{\lambda_i-i+j} - h_{\lambda_i-i-j})_{1\le i,j\le l}, \tag{16}$$

where h_n is the nth complete symmetric function of the form in equation (2). Define vertex operators

$$V_O(z) := (1 - z^2)e^{\xi(\mathbf{x},z)}e^{-\xi(\tilde{\partial}\mathbf{x},z^{-1})}e^{-\xi(\tilde{\partial}\mathbf{x},z)} \tag{17}$$

$$V_O^*(z) := e^{-\xi(\mathbf{x},z)}e^{\xi(\tilde{\partial}\mathbf{x},z^{-1})}e^{\xi(\tilde{\partial}\mathbf{x},z)} \tag{18}$$

and let

$$V_O(z) = \sum_{n\in\mathbb{Z}} V_n^O z^n, \quad V_O^*(z) = \sum_{n\in\mathbb{Z}} V_n^{O*} z^n.$$

Observe that this vertex operator $V_O(z)$ is the same as $V_\pi(z)$ for $\pi = (2)$ in [2].

The operator V_n^O is a raising operator of the orthogonal Schur function, i.e.,

$$S_\lambda^O(\mathbf{x}) = V_{\lambda_1}^O V_{\lambda_2}^O \cdots V_{\lambda_l}^O \cdot 1 \tag{19}$$

for a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$.

Define the generating functions

$$X^O(k) = \sum_{j\in\mathbb{Z}+\frac{1}{2}} X_j^O k^{-j-\frac{1}{2}} = V_O(k)e^K k^{H_0}, \tag{20}$$

$$X^{O*}(k) = \sum_{j\in\mathbb{Z}+\frac{1}{2}} X_j^{O*} k^{-j-\frac{1}{2}} = V_O^*(k)e^K k^{H_0}. \tag{21}$$

It can be checked that

$$\{X_i^O, X_j^O\} = 0,\ \{X_i^{O*},\ X_j^{O*}\} = 0,\ \{X_i^O,\ X_j^{O*}\} = \delta_{i+j,0}. \tag{22}$$

Definition 3.1. *For an unknown function $\tau = \tau(\mathbf{x})$, the bilinear equation*

$$\sum_{j\in\mathbb{Z}+\frac{1}{2}} X_j^{O*}\tau \otimes X_{-j}^O\tau = 0 \tag{23}$$

is called the orthogonal/symmplectic KP hierarchy, and denoted by OSKP hierarchy for short.

Equation (23) is equivalent to

$$\sum_{n+m=-1} V_n^{O*}\tau \otimes V_m^O\tau = 0. \tag{24}$$

It is obvious that equation (24) can be rewritten as

$$\frac{1}{2\pi i}\oint (1-z^2)e^{\xi(\mathbf{x}-\mathbf{x}',z)}dz\tau(\mathbf{x}'+[z^{-1}]+[z])\tau(\mathbf{x}-[z^{-1}]-[z]) = 0 \tag{25}$$

with $\mathbf{x} = (x_1, x_2, \cdots)$ and $\mathbf{x}' = (x_1', x_2', \cdots)$ being arbitrary parameters. Here the symbol $[z]$ denotes $(z, \frac{z^2}{2}, \frac{z^3}{3}, \cdots)$ and the integration means taking the coefficient of $\frac{1}{z}$ of the integrand in the formal Laurent series expansion in z. Then the equation (25) is equivalent to

$$\mathrm{Res}(1-z^2)e^{\xi(\mathbf{x}-\mathbf{x}',z)}\tau(\mathbf{x}'+[z^{-1}]+[z])\tau(\mathbf{x}-[z^{-1}]-[z]) = 0. \tag{26}$$

Let us replace $(\mathbf{x}', \mathbf{x})$ with $(\mathbf{x}+\mathbf{u}, \mathbf{x}-\mathbf{u})$ and consider the Taylor series expansion at $x' = x$, i.e., expand with respect to $\mathbf{u} = (u_1, u_2, \cdots)$. Hence, we obtain

$$\sum_{i-j+k=-1} P_i(-2\mathbf{u})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})\tau(\mathbf{x}+\mathbf{u})\tau(\mathbf{x}-\mathbf{u}) - \sum_{i-j+k=-3} P_i(-2\mathbf{u})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})\tau(\mathbf{x}+\mathbf{u})\tau(\mathbf{x}-\mathbf{u}) = 0. \tag{27}$$

By taking the coefficient of $\mathbf{u}^{\mathbf{n}} = u_1^{n_1}u_2^{n_2}\cdots$, we get many bilinear equations. Taking the coefficient of $1 = \mathbf{u}^{\mathbf{0}}$, we get

$$\sum_{k=0}^{\infty} P_{k+1}(D_{\mathbf{x}})P_k(D_{\mathbf{x}})\tau(\mathbf{x})\cdot\tau(\mathbf{x}) - \sum_{k=0}^{\infty} P_{k+3}(D_{\mathbf{x}})P_k(D_{\mathbf{x}})\tau(\mathbf{x})\cdot\tau(\mathbf{x}) = 0, \tag{28}$$

where $D_{\mathbf{x}} = (D_{x_1}, \frac{1}{2}D_{x_2}, \frac{1}{3}D_{x_3}, \cdots)$. We see that every differential equation with respect to $\mathbf{x}$ contained in the orthogonal KP hierarchy is of infinite order. This reflects the fact that the integrand of (25) with $\mathbf{x}' = \mathbf{x}$ may be singular not only at $z = 0$, but also at $z = \infty$.

For a Young diagram $\lambda = (\lambda_1, \cdots, \lambda_l)$, the symplectic Schur function[6,9] is defined to be

$$S_\lambda^{Sp} = \frac{1}{2}\det(h_{\lambda_i-i+j} + h_{\lambda_i-i-j+2})_{1\le i,j\le l},$$

where h_n is the nth complete symmetric function. The Symplectic symmetric function can be obtained by vertex operators as follows. Define the vertex operators

$$V_{Sp}(z) = e^{\xi(\mathbf{x},z)}e^{-\xi(\partial\mathbf{x},z^{-1})}e^{-\xi(\partial\mathbf{x},z)} \tag{29}$$

$$V_{Sp}^*(z) = (1-z^2)e^{-\xi(\mathbf{x},z)}e^{\xi(\partial\mathbf{x},z^{-1})}e^{\xi(\partial\mathbf{x},z)} \tag{30}$$

and let

$$V_{Sp}(z) = \sum_{n\in\mathbb{Z}} V_n^{Sp}z^n,\quad V_{Sp}^*(z) = \sum_{n\in\mathbb{Z}} V_n^{Sp*}z^n,$$

here the vertex operator $V_{Sp}(z)$ is the same as $V_\pi(z)$ for $\pi = (1^2)$ in [2].

The operator V_n^{Sp} is a raising operator of the symplectic Schur function, i.e.,

$$S_\lambda^{Sp}(\mathbf{x}) = S_\lambda^{Sp}(\mathbf{x}) = V_{\lambda_1}^{Sp}V_{\lambda_2}^{Sp}\cdots V_{\lambda_l}^{Sp}\cdot 1 \tag{31}$$

for a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$.

For an unknown function $\tau = \tau(\mathbf{x})$, the bilinear equation

$$\sum_{n+m=-1} V_n^{Sp*}\tau \otimes V_m^{Sp}\tau = 0 \tag{32}$$

gives the same integrable system as the bilinear equation (23), that is why we call this integrable system OSKP hierarchy.

4. ORTHOGONAL TYPE BOSON-FERMION CORRESPONDENCE

For Maya diagrams $|\mathbf{u}\rangle$ and $|\mathbf{v}\rangle$, the pairing $\langle\mathbf{v}|\mathbf{u}\rangle$ is defined by the formula

$$\langle\mathbf{v}|\mathbf{u}\rangle = \delta_{v_1+u_1,0}\delta_{v_2+u_2,0}\cdots.$$

Define operators H_n by

$$H_n = \sum_{j\in\mathbb{Z}+1/2} :\psi_{-j}\psi^*_{j+n}:$$

and $H(x) = \sum_{n=1}^{\infty} x_n H_n$.

From the actions of Fermions on Maya diagrams, we get the action of H_1 on a Maya diagram is H_1 sending a Maya diagram $|\mathbf{u}\rangle$ to the sum over all Maya diagrams which can be obtained from $|\mathbf{u}\rangle$ by moving a black stone to the right. We define P_n and Q_n from equations

$$\exp\left(\sum_{m\geq 1}\frac{H_m}{m}k^m\right) = \sum_{n\geq 0} Q_{(n)}k^n,\quad \exp\left(\sum_{m\geq 1}\frac{H_{-m}}{m}k^m\right) = \sum_{n\geq 0} P_{(n)}k^n \tag{33}$$

The action of $Q_{(m)}$ on Maya diagram is defined by $Q_{(m)}$ sending the Maya diagram $|\mathbf{u}\rangle$ to the sum over all Maya diagrams which can be obtained from $|\mathbf{u}\rangle$ by moving black stones m times to the right and no one black stone is moved twice. Then, $Q_{(1^m)}$ sends Maya diagram $|\mathbf{u}\rangle$ to the sum over all Maya diagrams which can be obtained from $|\mathbf{u}\rangle$ by moving black stones m times to the right and no two adjacent black stones move at the same time.

Define

$$\psi_j^O = \sum_{n=1}^{\infty}(-1)^n\psi_{n+j}Q_{1^n} - \sum_{n=1}^{\infty}(-1)^n\psi_{n+j+2}Q_{1^n}, \tag{34}$$

$$\psi_j^{O*} = \sum_{n=1}^{\infty}\psi_{n+j}Q_n. \tag{35}$$

The actions of ψ_j^O, ψ_j^{O*}, where $j \in \frac{1}{2}+\mathbb{Z}$, on Maya diagram can be obtained from the actions of ψ_j, ψ_j^* and $Q_{(m)},\ Q_{(1^m)}$ on Maya diagram according to (34-35).

Let λ be a Young diagram, and λ' be its conjugate. The Frobenius notation $\lambda = (n_1,\cdots,n_l|m_1,\cdots,m_l)$ describes the Young diagram λ by $n_i = \lambda_i - i$, $m_i = \lambda'_i - i$, where l is the number of the boxes in the NW-SE diagonal line of λ.

Under the Boson-Fermion correspondence, the basis vector

$$\psi_{n_1}\cdots\psi_{n_l}\psi^*_{m_1}\cdots\psi^*_{m_l}|\text{vac}\rangle \text{ for } n_1 < \cdots < n_l < 0 \text{ and } m_1 < \cdots < m_l < 0$$

of Fermionic Fock space of charge zero goes over into the Schur function S_λ multiplied by $a_\lambda = (-1)^{\sum_{i=1}^{l}(m_i+\frac{1}{2})+\frac{l(l-1)}{2}}$, where $\lambda = (-n_1 - \frac{1}{2},\cdots,-n_l-\frac{1}{2}|-m_1-\frac{1}{2},\cdots,-m_l-\frac{1}{2})$, i.e.,

$$S_\lambda = a_\lambda\langle\text{vac}|e^{H(x)}\psi_{n_1}\cdots\psi_{n_l}\psi^*_{m_1}\cdots\psi^*_{m_l}|\text{vac}\rangle, \tag{36}$$

then we have

Proposition 4.1. *For $\lambda = (-n_1-\frac{1}{2},\cdots,-n_l-\frac{1}{2}|-m_1-\frac{1}{2},\cdots,-m_l-\frac{1}{2})$, the orthogonal Schur function S^O_λ is obtained from*

$$S^O_\lambda = (-1)^{\sum_{i=1}^{l}(m_i+\frac{1}{2})+\frac{l(l-1)}{2}}\langle vac|e^{H(x)}\psi^O_{n_1}\cdots\psi^O_{n_l}\psi^{O*}_{m_1}\cdots\psi^{O*}_{m_l}|vac\rangle. \tag{37}$$

Using the Fermions ψ_j and ψ_j^*, we can also get the orthogonal Schur function by the following formula.

Proposition 4.2. *For $\lambda = (-n_1-\frac{1}{2},\cdots,-n_l-\frac{1}{2}|-m_1-\frac{1}{2},\cdots,-m_l-\frac{1}{2})$, the orthogonal Schur function S^O_λ is obtained from*

$$S^O_\lambda = (-1)^{\sum_{i=1}^{l}(m_i+\frac{1}{2})+\frac{l(l-1)}{2}}\langle vac|e^{H(x)}e^{-\sum_{n=1}^{\infty}\frac{1}{2n}(H_n^2+H_{2n})}\psi_{n_1}\cdots\psi_{n_l}\psi^*_{m_1}\cdots\psi^*_{m_l}|vac\rangle. \tag{38}$$

We can get the symplectic Schur function similarly.

Proposition 4.3. *For $\lambda = (-n_1-\frac{1}{2},\cdots,-n_l-\frac{1}{2}|-m_1-\frac{1}{2},\cdots,-m_l-\frac{1}{2})$, the symplectic Schur function S^{Sp}_λ is obtained from*

$$S^{Sp}_\lambda = a_\lambda\langle vac|e^{H(x)}\psi^{Sp}_{n_1}\cdots\psi^{Sp}_{n_l}\psi^{Sp*}_{m_1}\cdots\psi^{Sp*}_{m_l}|vac\rangle \tag{39}$$

$$= a_\lambda\langle vac|e^{H(x)}e^{-\sum_{n=1}^{\infty}\frac{1}{2n}(H_n^2-H_{2n})}\psi_{n_1}\cdots\psi_{n_l}\psi^*_{m_1}\cdots\psi^*_{m_l}|vac\rangle, \tag{40}$$

where $a_\lambda = (-1)^{\sum_{i=1}^{l}(m_i+\frac{1}{2})+\frac{l(l-1)}{2}}$.

For example, we can calculate $S^O_{\square\square}$ in the following two ways. The first way is

$$\psi^O_{-\frac{3}{2}} = \psi_{-\frac{3}{2}} - \psi_{\frac{1}{2}} - \psi_{-\frac{1}{2}}Q + \psi_{\frac{3}{2}}Q + \cdots,$$
$$\psi^{O*}_{-\frac{1}{2}} = \psi^*_{-\frac{1}{2}} + \psi^*_{\frac{1}{2}}Q + \psi^*_{\frac{3}{2}}Q_2 + \cdots,$$

Then,

$$\begin{aligned} S^O_{\square\square} &= \langle \mathrm{vac}|e^{H(x)}\psi^O_{-\frac{3}{2}}\psi^{O*}_{-\frac{1}{2}}|\mathrm{vac}\rangle \\ &= \langle \mathrm{vac}|e^{H(x)}(\psi_{-\frac{3}{2}} - \psi_{\frac{1}{2}})\psi^*_{-\frac{1}{2}}|\mathrm{vac}\rangle \\ &= S_{\square\square} - S_0. \end{aligned}$$

In the second way, we know that for the Maya diagram

$$|\gamma\rangle = \begin{matrix} \cdots & \circ & \circ & \circ & \bullet & \bullet & \circ & \bullet & \bullet & \bullet & \cdots \\ & -\frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \end{matrix}$$

we have $H_m\gamma = 0$ when $m > 2$. Then

$$\begin{aligned} S^O_{\square\square} &= \langle \mathrm{vac}|e^{H(x)}e^{-\sum_{n=1}^{\infty}\frac{1}{2n}(H_n^2+H_{2n})}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|\mathrm{vac}\rangle \\ &= \langle \mathrm{vac}|e^{H(x)}(1 - \frac{1}{2}(H_1^2 + H_2))|\gamma\rangle \\ &= \langle \mathrm{vac}|e^{H(x)}(1 - Q_2)|\gamma\rangle \\ &= S_{\square\square} - S_0. \end{aligned}$$

Then, we obtain the orthogonal type Boson-Fermion correspondence.

Proposition 4.4. *The Fermions ψ^O_j, ψ^{O*}_j are realized in the Bosonic Fock space by X^O_j, X^{O*}_j, i.e., for any $|u\rangle \in \mathcal{F}$, we have*

$$\langle l|e^{H(x)}\psi^O_j|u\rangle = X^O_j\langle l|e^{H(x)}|u\rangle, \quad \langle l|e^{H(x)}\psi^{O*}_j|u\rangle = X^{O*}_j\langle l|e^{H(x)}|u\rangle, \tag{41}$$

where $\langle l| = \langle \cdots, l-\frac{5}{2}, l-\frac{3}{2}, l-\frac{1}{2}|$.

5. THE MODIFIED ORTHOGONAL KP HIERARCHY

Now, we consider the functional relations for a sequence of τ-functions connected by successive application of vertex operators. Let $\tau_0 := \tau(\mathbf{x})$ be a solution of the orthogonal KP hierarchy (23). Let $\tau_1 := V_O(\alpha)\tau$ and $\tau_1' = V_{Sp}(\alpha)\tau$ with an arbitrary constant $\alpha \in \mathbb{C}^\times$. Then τ_1 and τ_1' are also solutions of (23) . Moreover, we can deduce the bilinear equation

$$\sum_{n+m=-2} V^{O*}_n\tau_n \otimes V^O_n\tau_{n+1} = 0$$

from (23) multiplied by $1 \otimes V_O(\alpha)$ or

$$\sum_{n+m=-2} V^{Sp*}_n\tau_n \otimes V^{Sp}_n\tau_{n+1} = 0$$

from (32) multiplied by $1 \otimes V_{Sp}(\alpha)$. The two equations above can be equivalently rewritten into the equation

$$\frac{1}{2\pi i}\oint z(1-z^2)e^{\xi(\mathbf{x}-\mathbf{x}',z)}dz\tau_n(\mathbf{x}' + [z^{-1}] + [z])\tau_{n+1}(\mathbf{x} - [z^{-1}] - [z]) = 0. \tag{42}$$

Replace $(\mathbf{x}', \mathbf{x})$ with $(\mathbf{x}+\mathbf{u}, \mathbf{x}-\mathbf{u})$ and consider the Taylor series expansion at $x' = x$, we obtain

$$\sum_{i-j+k=-2} P_i(-2\mathbf{u})P_j(\tilde{\partial}_\mathbf{u})P_k(\tilde{\partial}_\mathbf{u})\tau_n(\mathbf{x}+\mathbf{u})\tau_{n+1}(\mathbf{x}-\mathbf{u}) + \sum_{i-j+k=-4} P_i(-2\mathbf{u})P_j(\tilde{\partial}_\mathbf{u})P_k(\tilde{\partial}_\mathbf{u})\tau_n(\mathbf{x}+\mathbf{u})\tau_{n+1}(\mathbf{x}-\mathbf{u}) = 0. \tag{43}$$

By taking the coefficient of $\mathbf{u}^\mathbf{n} = u_1^{n_1}u_2^{n_2}\cdots$ for variety n, we will get many bilinear equations. Taking the coefficient of $1 = \mathbf{u}^\mathbf{0}$, we get

$$\sum_{k=0}^{\infty} P_{k+2}(D_\mathbf{x})P_k(D_\mathbf{x})\tau_n(\mathbf{x}) \cdot \tau_{n+1}(\mathbf{x}) - \sum_{k=0}^{\infty} P_{k+3}(D_\mathbf{x})P_k(D_\mathbf{x})\tau_n(\mathbf{x}) \cdot \tau_{n+1}(\mathbf{x}) = 0. \tag{44}$$

6. UNIVERSAL CHARACTER AND UC HIERARCHY

For a pair of Young diagrams $\lambda = (\lambda_1, \cdots, \lambda_l)$ and $\mu = (\mu_1, \cdots, \mu_{l'})$, we define the universal character as a polynomial in $\mathbf{x} = (x_1, x_2, \cdots)$ and $\mathbf{y} = (y_1, y_2, \cdots)$:

$$S^O_{[\lambda,\mu]}(\mathbf{x},\mathbf{y}) = (-1)^{ll'+\frac{l'(l'+1)}{2}} \times \det\begin{pmatrix} h_{\mu_{l'-i+1}-l-l'+i-j-1}(\mathbf{y}) - h_{\mu_{l'-i+1}-l-l'+i+j-1}(\mathbf{y}), & 1 \le i \le l' \\ h_{\lambda_{i-l'}+l'-i+j}(\mathbf{x}) - h_{\lambda_{i-l'}+l'-i-j}(\mathbf{x}), & l'+1 \le i \le l+l' \end{pmatrix}. \tag{45}$$

We can see that the orthogonal Schur function $S^O_\lambda(\mathbf{x})$ is a special case of the universal character: $S^O_\lambda(\mathbf{x}) = S^O_{[\lambda,\varnothing]}(\mathbf{x},\mathbf{y})$.

Let us introduce the vertex operators

$$X^+(k) = (1-k^2)e^{\xi(\mathbf{x},k)}e^{-\xi(\tilde{\partial}_\mathbf{y},k^{-1})}e^{-\xi(\tilde{\partial}_\mathbf{y},k)}e^{-\xi(\tilde{\partial}_\mathbf{x},k^{-1})}e^{-\xi(\tilde{\partial}_\mathbf{x},k)}, \tag{46}$$

$$X^-(k) = e^{-\xi(\mathbf{x},k)}e^{\xi(\tilde{\partial}_\mathbf{y},k^{-1})}e^{\xi(\tilde{\partial}_\mathbf{y},k)}e^{\xi(\tilde{\partial}_\mathbf{x},k^{-1})}e^{\xi(\tilde{\partial}_\mathbf{x},k)}, \tag{47}$$

$$Y^+(k) = (1-k^2)e^{\xi(\mathbf{y},k)}e^{-\xi(\tilde{\partial}_\mathbf{x},k^{-1})}e^{-\xi(\tilde{\partial}_\mathbf{x},k)}e^{-\xi(\tilde{\partial}_\mathbf{y},k)}e^{-\xi(\tilde{\partial}_\mathbf{y},k^{-1})}, \tag{48}$$

$$Y^-(k) = e^{-\xi(\mathbf{y},k^{-1})}e^{\xi(\tilde{\partial}_\mathbf{x},k^{-1})}e^{\xi(\tilde{\partial}_\mathbf{x},k)}e^{\xi(\tilde{\partial}_\mathbf{y},k)}e^{\xi(\tilde{\partial}_\mathbf{y},k^{-1})}, \tag{49}$$

and let $X^\pm(k) = \sum_{n\in\mathbb{Z}} X^\pm_n k^n$, $Y^\pm(k) = \sum_{n\in\mathbb{Z}} Y^\pm_n k^n$.

It can be checked that the $X^\pm_n$ satisfy the formionic relations: $X^\pm_n X^\pm_{m+1} + X^\pm_{n+1}X^\pm_n = 0$ and $X^+_{n+1}X^-_m + X^-_{m+1}X^+_n = \delta_{n+m+1,0}$. The same relations hold also for $Y^\pm_n$. Moreover, $X^\pm_n$ and $Y^\pm_n$ mutually commute.

Proposition 6.1. *The universal character $S^O_{[\lambda,\mu]}(\mathbf{x},\mathbf{y})$ can be obtained by means of these operators:*

$$S^O_{[\lambda,\mu]}(\mathbf{x},\mathbf{y}) = X^+_{\lambda_1}\cdots X^+_{\lambda_l}Y^+_{\mu_1}\cdots Y^+_{\mu_l}\cdot 1. \tag{50}$$

Proof. We will use the Vandermonde-like identity,

$$\det(k_i^{l-j} - k_i^{l+j}) = \prod_{1\le i<j\le l}(k_i - k_j)(1 - k_ik_j).$$

Then,

$$\begin{aligned}
&X^+(k_1)\cdots X^+(k_l)Y^+(w_1^{-1})\cdots Y^+(w_l'^{-1})\cdot 1 \\
&= \prod_{i=1}^{l}(1-k_i^2)\prod_{j=1}^{l'}(1-\frac{1}{w_i^2})\prod_{i<j}(1-k_ik_j)(1-\frac{k_j}{k_i})\prod_{i,k}(1-\frac{1}{k_iw_j})(1-\frac{k_i}{w_j}) \\
&\quad\times\prod_{a<b}(1-\frac{w_b}{w_a})(1-\frac{1}{w_aw_b})e^{\xi(\mathbf{x},k_1)}\cdots e^{\xi(\mathbf{x},k_l)}e^{\xi(\mathbf{y},w_1^{-1})}\cdots e^{\xi(\mathbf{y},w_{l'}^{-1})} \\
&= (-1)^{ll'+\frac{l'(l'+1)}{2}}\prod_i k_{l-i}^{-(l+2l'-i)}\prod_j w_{l'-i+1}^{-(2l+2l'-i+1)}\det\begin{pmatrix} w_{l'-i+1}^{l+l'-j} - w_{l'-i+1}^{l+l'+j}, & 1\le i\le l' \\ k_{i-l'}^{l+l'-j} - k_{i-l'}^{l+l'+j}, & l'+1\le i\le l+l'\end{pmatrix} \\
&\quad\times e^{\xi(\mathbf{x},k_1)}\cdots e^{\xi(\mathbf{x},k_l)}e^{\xi(\mathbf{y},w_1^{-1})}\cdots e^{\xi(\mathbf{y},w_{l'}^{-1})} \\
&= (-1)^{ll'+\frac{l'(l'+1)}{2}}\det\begin{pmatrix} w_{l'-i+1}^{-l-l'+i-j-1} - w_{l'-i+1}^{-l-l'+i+j-1}, & 1\le i\le l' \\ k_{i-l'}^{-l'+i-j} - k_{i-l'}^{-l'+i+j}, & l'+1\le i\le l+l'\end{pmatrix} \\
&\quad\times e^{\xi(\mathbf{x},k_1)}\cdots e^{\xi(\mathbf{x},k_l)}e^{\xi(\mathbf{y},w_1^{-1})}\cdots e^{\xi(\mathbf{y},w_{l'}^{-1})}.
\end{aligned}$$

Taking the coefficient of $k_1^{\lambda_1}\cdots k_l^{\lambda_l}w_1^{-\mu_1}\cdots w_{l'}^{-\mu_l}$, we will get (50). □

We give a remark here to explain the difference between the universal characters $S^O_{[\lambda,\mu]}(\mathbf{x},\mathbf{y})$ here and that in our paper [4]. The vertex operators which realize $S^O_{[\lambda,\mu]}(\mathbf{x},\mathbf{y})$ in this paper are more complex than that in [4], but in this paper, the universal characters $S^O_{[\lambda,\mu]}(\mathbf{x},\mathbf{y})$ can be described by the determinant, that in [4] can not described by determinant.

Now we can define a UC hierarchy where UC is the abbreviation of universal character.

Definition 6.2. *For an unknown function $\tau = \tau(\mathbf{x},\mathbf{y})$, the system of bilinear relations*

$$\sum_{n+m=-1} X^-_n\tau\otimes X^+_m\tau = \sum_{n+m=-1} Y^-_n\tau\otimes Y^+_m\tau = 0 \tag{51}$$

is called the orthogonal UC hierarchy.

If $\tau = \tau(\mathbf{x}, \mathbf{y})$ does not depend on $\mathbf{y} = (y_1, y_2, \cdots)$, the second equality of (51) trivially holds and the first one is reduced to the bilinear expression (23) of the OSKP hierarchy. From this aspect, we treat the orthogonal UC hierarchy as an extension of the OSKP hierarchy.

It is obvious that (51) can be rewritten into the form

$$\frac{1}{2\pi i}\oint (1-z^2)e^{\xi(\mathbf{x}'-\mathbf{x}'',z)}dz \quad \tau(\mathbf{x}''+[z^{-1}]+[z],\mathbf{y}''+[z])\cdot\tau(\mathbf{x}'-[z^{-1}]-[z],\mathbf{y}'-[z]) = 0, \tag{52}$$

$$\frac{1}{2\pi i}\oint (1-w^2)e^{\xi(\mathbf{y}'-\mathbf{y}'',w)}dw \quad \tau(\mathbf{x}''+[w],\mathbf{y}''+[w^{-1}]+[w])\cdot\tau(\mathbf{x}'-[w],\mathbf{y}'-[w^{-1}]-[w]) \ = 0 \tag{53}$$

for arbitrary $\mathbf{x}, \mathbf{x}', \mathbf{y}$ and $\mathbf{y}'$. Consider their Taylor expansions at $(\mathbf{x} = \mathbf{x}', \mathbf{y} = \mathbf{y}')$, that is, replacing $(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}')$ with $(\mathbf{x}-\mathbf{u}, \mathbf{x}+\mathbf{u}, \mathbf{y}-\mathbf{v}, \mathbf{y}+\mathbf{v})$ and expand with respect to $(\mathbf{u}, \mathbf{v}) = (u_1, u_2, \cdots, v_1, v_2, \cdots)$, then we get

$$\begin{aligned}&\sum_{i-j+k-m+n=-1} P_i(-2\mathbf{u})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})P_m(\tilde{\partial}_{\mathbf{u}})P_n(\tilde{\partial}_{\mathbf{u}})\tau(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})\tau(\mathbf{x}-\mathbf{u},\mathbf{y}-\mathbf{v})\\ &-\sum_{i-j+k-m+n=-3} P_i(-2\mathbf{u})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})P_m(\tilde{\partial}_{\mathbf{u}})P_n(\tilde{\partial}_{\mathbf{u}})\tau(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})\tau(\mathbf{x}-\mathbf{u},\mathbf{y}-\mathbf{v}) = 0.\end{aligned}$$

and

$$\begin{aligned}&\sum_{i-j+k-m+n=-1} P_i(-2\mathbf{v})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})P_m(\tilde{\partial}_{\mathbf{u}})P_n(\tilde{\partial}_{\mathbf{u}})\tau(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})\tau(\mathbf{x}-\mathbf{u},\mathbf{y}-\mathbf{v})\\ &-\sum_{i-j+k-m+n=-3} P_i(-2\mathbf{v})P_j(\tilde{\partial}_{\mathbf{u}})P_k(\tilde{\partial}_{\mathbf{u}})P_m(\tilde{\partial}_{\mathbf{u}})P_n(\tilde{\partial}_{\mathbf{u}})\tau(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v})\tau(\mathbf{x}-\mathbf{u},\mathbf{y}-\mathbf{v}) = 0.\end{aligned}$$

Taking the coefficient of $\mathbf{u^n v^m}$ leads to many differential equation with respect to $\mathbf{x}, \mathbf{y}$, these differential equations are all of infinite order. This reflects that the integrands above with $(\mathbf{x} = \mathbf{x}', \mathbf{y} = \mathbf{y}')$ may be singular not only at $z = 0, w = 0$ but also at $z = \infty, w = \infty$.

In the follows, we give a class of polynomial solutions of the orthogonal UC hierarchy. From the relations between $X_n^{\pm}$, $Y_n^{\pm}$, we obtain

$$\left(\sum_{n+m=-1} X_n^- \otimes X_m^+\right)(X_t^+\otimes X_t^+) = (X_{t+1}^+\otimes X_{t-1}^+)\left(\sum_{n+m=-1} X_n^- \otimes X_m^+\right) \tag{54}$$

$$\left(\sum_{n+m=-1} Y_n^- \otimes Y_m^+\right)(X_t^+\otimes X_t^+) = (X_t^+\otimes X_t^+)\left(\sum_{n+m=-1} Y_n^- \otimes Y_m^+\right) \tag{55}$$

that is, if $\tau = \tau(\mathbf{x}, \mathbf{y})$ is a solution of (51), so is $X_t^+\tau$, we can verify in the same way that $Y_t^+\tau$ is also a solution of (51). By equation (50), we obtain

Proposition 6.3. *All the universal characters $S^O_{[\lambda,\mu]}(\mathbf{x}, \mathbf{y})$ are solutions of the orthogonal UC hierarchy.*

It is known that if $\tau = \tau(\mathbf{x}, \mathbf{y})$ is a solution of (51), so are $X^+(\alpha)\tau$ and $Y^+(\beta)\tau$ for arbitrary constants $\alpha, \beta \in \mathbb{C}^\times$. Then we will consider the bilinear relations among the solutions connected by the vertex operators. The modified orthogonal UC hierarchy is introduced as follows.

Definition 6.4. *Suppose $\tau_{m,n} = \tau_{m,n}(\mathbf{x}, \mathbf{y})$ is a solution of the orthogonal UC hierarchy (51). Let*

$$\tau_{m+1,n} = X^+(\alpha_m)\tau_{m,n}, \tau_{m,n+1} = Y^+(\beta_n)\tau_{m,n},$$
$$\tau_{m+1,n+1} = X^+(\alpha_m)Y^+(\beta_n)\tau_{m,n} = Y^+(\beta_n)X^+(\alpha_m)\tau_{m,n}$$

for arbitrary constants $\alpha_m, \beta_n \in \mathbb{C}^\times$. From equation (51), we can get the equations satisfied by $\tau_{m,n}$'s, which are called the modified orthogonal UC hierarchy.

For τ- function $\tau_{m,n} = \tau_{m,n}(\mathbf{x}, \mathbf{y})$, the modified orthogonal UC hierarchy includes the following bilinear equations:

$$\sum_{i+j=-2} X_i^-\tau_{m,n}\otimes X_j^+\tau_{m+1,n} = \sum_{i+j=-1} Y_i^-\tau_{m,n}\otimes Y_j^+\tau_{m+1,n} = 0, \tag{56}$$

$$\tau_{m,n}\otimes\tau_{m+1,n+1} - \sum_{i+j=0} X_i^-\tau_{m+1,n}\otimes X_j^+\tau_{m,n+1} = \sum_{i+j=-2} Y_i^-\tau_{m+1,n}\otimes Y_j^+\tau_{m,n+1} = 0. \tag{57}$$

Here the first equation and the second equation can be deduced from (51) by applying $1\otimes X^+(\alpha_m)$ and $X^+(\alpha_m)\otimes Y^+(\beta_n)$, respectively.

From the definition of $\tau_{m,n}$, for a solution $\tau_{0,0}$ of the orthogonal UC hierarchy, we have

$$\tau_{m,n} = \prod_{i=0}^{m-1} X^+(\alpha_i) \prod_{j=0}^{n-1} Y^+(\beta_j)\tau_{0,0}, \tag{58}$$

where

$$\prod_{i=0}^{m-1} X^+(\alpha_i) = X^+(\alpha_{m-1})\cdots X^+(\alpha_1)X^+(\alpha_0),$$

then we have the following bilinear equations.

Proposition 6.5. *For integers $m, n \geq 0$, it holds that*

$$\sum_{i+j=-m-1} X_i^- \tau_{0,0} \otimes X_j^+ \tau_{m,n} = \sum_{i+j=-n-1} Y_i^- \tau_{0,0} \otimes Y_j^+ \tau_{m,n} = 0, \tag{59}$$

$$\tau_{0,0} \otimes \tau_{1,n} - \sum_{i+j=0} X_i^- \tau_{1,0} \otimes X_j^+ \tau_{0,n} = \sum_{i+j=-n-1} Y_i^- \tau_{1,0} \otimes Y_j^+ \tau_{0,n} = 0, \tag{60}$$

$$\sum_{i+j=-m-1} X_i^- \tau_{0,1} \otimes X_j^+ \tau_{m,0} = \tau_{0,0} \otimes \tau_{m,1} - \sum_{i+j=0} Y_i^- \tau_{0,1} \otimes Y_j^+ \tau_{m,0} = 0. \tag{61}$$

The results above are obtained by applying $1 \otimes \prod_{i=0}^{m-1} X^+(\alpha_i) \prod_{j=0}^{n-1} Y^+(\beta_j)$, $X^+(\alpha_0) \otimes \prod_{j=0}^{n-1} Y^+(\beta_j)$ and $Y^+(\beta_0) \otimes \prod_{i=0}^{m-1} X^+(\alpha_i)$ to (51).

Let us look closely at (59), which corresponds to the orthogonal UC hierarchy (51) when $m = n = 0$. It can be equivalently rewritten into

$$\frac{1}{2\pi i}\oint z^m(1-z^2)e^{\xi(\mathbf{x}-\mathbf{x}',z)}dz \quad \tau_{0,0}(\mathbf{x}'+[z^{-1}]+[z], \mathbf{y}'+[z^{-1}]+[z]) \cdot \tau_{m,n}(\mathbf{x}-[z^{-1}]-[z], \mathbf{y}-[z^{-1}]-[z]) = 0, \tag{62}$$

$$\frac{1}{2\pi i}\oint w^n(1-w^2)e^{\xi(\mathbf{y}-\mathbf{y}',w)}dw \quad \tau_{0,0}(\mathbf{x}'+[w^{-1}]+[w], \mathbf{y}'+[w^{-1}]+[w]) \cdot \tau_{m,n}(\mathbf{x}-[w^{-1}]-[w], \mathbf{y}-[w^{-1}]-[w]) = 0. \tag{63}$$

Let $I, J \subset \mathbb{Z}$ be a disjoint pair of finite indexing sets. By specializing the parameters in (62) and (63) as

$$\mathbf{x}' = \mathbf{x} - \sum_{j\in I}[t_i] + \sum_{j\in J}[t_j],\ \mathbf{y}' = \mathbf{y} - \sum_{j\in I}[t_i^{-1}] + \sum_{j\in J}[t_j^{-1}],$$

we get

$$\Omega_1 := z^m(1-z^2)e^{\xi(\mathbf{x}-\mathbf{x}',z)}dz = z^m(1-z^2)\frac{\prod_{j\in J}(1-t_jz)}{\prod_{j\in I}(1-t_jz)}dz,$$

$$\Omega_2 := w^n(1-w^2)e^{\xi(\mathbf{x}-\mathbf{x}',w)}dw = w^n(1-w^2)\frac{\prod_{j\in J}(1-w/t_j)}{\prod_{j\in I}(1-w/t_j)}dw.$$

Let $z = 1/w$, we find that

$$\Omega_2 = z^{|I|-|J|-m-n-4}\frac{\prod_{j\in I}(-t_j)}{\prod_{j\in J}(-t_j)}\Omega_1$$

Consequently, the integrands of (62) and (63) coincide up to constant functor if the condition $|I| - |J| = m + n + 4$ holds. Let

$$F(z) = z^m(1-z^2)\frac{\prod_{j\in J}(1-t_jz)}{\prod_{j\in I}(1-t_jz)} \cdot \tau(\mathbf{x}'+[z^{-1}]+[z], \mathbf{y}'+[z^{-1}]+[z]) \cdot \tau(\mathbf{x}-[z^{-1}]-[z], \mathbf{y}-[z^{-1}]-[z])$$

in the integrand of (62), hence, we get

$$\int_{C_1} F(z)dz = \int_{C_2} F(z)dz = 0,$$

where C_1 and C_2 are a positively oriented small circle around $z = 0$ and $z = \infty$ respectively such that all the other singularities are out of it. Then, we obtain

$$\sum_{i\in I} \mathrm{Res}_{z=1/t_i} F(z)dz = 0. \tag{64}$$

This means that the residue calculus at possible essential singularities $z = 0, \infty$ is avoided for the presence of two bilinear equations (62) and (63).

For a function $f = f(\mathbf{x}, \mathbf{y})$, we define a shift operator T_i by

$$T_i(f) = f(\mathbf{x} - [t_i], y - [t_i^{-1}])$$

and $T_{\{i_1,\cdots,i_r\}} := T_{i_1} \cdots T_{i_r}(f)$ for short. Then (64) gives

$$\sum_{i\in I} t_i^n (1 - \frac{1}{t_i^2}) \frac{\prod_{j\in J}(t_i - t_j)}{\prod_{j\in I/\{i\}}(t_i - t_j)} T_{I/\{i\}}\tau_{0,0}(\mathbf{x} + [t_i^{-1}], \mathbf{y} + [t_i]) T_{J\cup\{i\}}\tau_{m,n}(\mathbf{x} - [t_i^{-1}], \mathbf{y} - [t_i]) = 0,$$

which can be regarded as a difference equation with each t_i being the difference interval. Then, we have

Proposition 6.6. *The following equations hold:*

1. *If $|I| - |J| = m + n + 4$ and $m, n \geq 0$, then*

$$\sum_{i\in I} t_i^n (t_i^2 - 1) \frac{\prod_{j\in J}(t_i - t_j)}{\prod_{j\in I/\{i\}}(t_i - t_j)} T_{I/\{i\}}\tau_{0,0}(\mathbf{x} + [t_i^{-1}], \mathbf{y} + [t_i]) T_{J\cup\{i\}}\tau_{m,n}(\mathbf{x} - [t_i^{-1}], \mathbf{y} - [t_i]) = 0.$$

2. *If $|I| - |J| = n + 3$ and $n \geq 0$, then*

$$T_I(\tau_{0,0})T_J(\tau_{1,n}) = \sum_{i\in I} (1 - \frac{1}{t_i^2}) \frac{\prod_{j\in J}(1 - t_j/t_i)}{\prod_{j\in I/\{i\}}(1 - t_j/t_i)} T_{I/\{i\}}\tau_{1,0}(\mathbf{x} + [t_i^{-1}], \mathbf{y} + [t_i]) \cdot T_{J\cup\{i\}}\tau_{0,n}(\mathbf{x} - [t_i^{-1}], \mathbf{y} - [t_i]) = 0.$$

3. *If $|I| - |J| = m + 3$ and $m \geq 0$, then*

$$T_I(\tau_{0,0})T_J(\tau_{m,1}) = \sum_{i\in I} (1 - \frac{1}{t_i^2}) \frac{\prod_{j\in J}(1 - t_i/t_j)}{\prod_{j\in I/\{i\}}(1 - t_i/t_j)} T_{I/\{i\}}\tau_{1,0}(\mathbf{x} + [t_i^{-1}], \mathbf{y} + [t_i]) \cdot T_{J\cup\{i\}}\tau_{0,n}(\mathbf{x} - [t_i^{-1}], \mathbf{y} - [t_i]) = 0.$$

Let $m = 1, n = 0, I = 1, 2, 3, 4, J = \emptyset$, and let $t_3 = t_1^{-1},\ t_4 = t_2^{-1}$, the first equation in Proposition 6.6 reduces to

$$(1 - t_1t_2)(t_2 - t_1)\tilde{T}_{12}(\tau_{0,0})\tau_{1,1} = t_2(t_1^2 + 1)\tilde{T}_2(\tau_{1,0})\tilde{T}_1(\tau_{0,1}) - t_1(t_2^2 + 1)\tilde{T}_1(\tau_{1,0})\tilde{T}_2(\tau_{0,1}), \tag{65}$$

where the notation $\tilde{T}_i$ is a shift operator defined by

$$\tilde{T}_i f(\mathbf{x}, \mathbf{y}) = f(\mathbf{x} - [t_i] - [t_i^{-1}], \mathbf{y} - [t_i] - [t_i^{-1}]).$$

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the support of Professors Ke Wu, Zi-Feng Yang and Shi-Kun Wang. This work is supported by the National Natural Science Foundation of China under Grant No. 11505046.

REFERENCES

[1] E. Date, M. Kashiwara, M. Jimbo, T. Miwa, Transformation groups for soliton equations, in: M. Jimbo, T. Miwa (eds.), Nonlinear integrable systems-classical theory and quantum theory (Kyoto, 1981), World Sci. Publishing, Singapore, 1983, pp. 39–119.
[2] B. Fauser, P.D. Jarvis, R.C. King, Plethystic vertex operators and Boson-Fermion correspondences. J. Phys. A: Math. Theor. 49 (2016), 425201.
[3] W. Fulton, J. Harris, Representation theory, A first course, Springer-Verlag, New York, 1991.
[4] F. Huang, N. Wang, Generalized symplectic Schur functions and SUC hierarchy, J. Math. Phys. 61 (2020), 061508.
[5] M. Jimbo, T. Miwa, Solitons and infinite dimensional Lie algebras, Publ. Res. Inst. Math. Sci. 19 (1983), 943–1001.
[6] N. Jing, B. Nie, Vertex operators, Weyl determinant formulae and Littlewood duality, Ann. Combin. 19 (2015), 427–442.
[7] N. Jing, N. Rozhkovskaya, Vertex operators arising from Jacobi–Trudi identities, Commun. Math. Phys. 346 (2016), 679–701.
[8] K. Koike, On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters, Adv. Math. 74 (1989), 57–86.

[9] K. Koike, I. Terada, Young-diagrammatic methods for the representation theory of the classical groups of types B_n, C_n, D_n, J. Algebra 107 (1987), 466–511.
[10] I.G. Macdonald, Symmetric functions and Hall polynomials, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 1979.
[11] T. Miwa, M. Jinbo, M. Jimbo, E. Date, Solitons: differential equations, symmetries and infinite dimensional algebras, Cambridge University Press, Cambridge, 2000.
[12] T. Tsuda, From KP/UC hierarchies to Painlevé equations, arXiv:1004.1347.
[13] T. Tsuda, Universal characters and an extension of the KP hierarchy, Commun. Math. Phys. 248 (2004), 501–526.

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics
Vol. **28(3)**; September (2021), *pp.* 303–308
DOI: https://doi.org/10.2991/jnmp.k.210419.002; ISSN 1402-9251; eISSN 1776-0852
https://www.atlantis-press.com/journals/jnmp

Research Article

A New Case of Separability in a Quartic Hénon-Heiles System

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ARTICLE INFO

Article History

Received 17 November 2020
Accepted 09 April 2021

Keywords

Integrable systems
separation of coordinates
integration in quadratures

2020 Mathematics Subject Classification

37J35
34M55

ABSTRACT

There are four quartic integrable Hénon-Heiles systems. Only one of them has been separated in the generic form while the other three have been solved only for particular values of the constants. We consider two of them, related by a canonical transformation, and we give their separation coordinates in a new case.

1. INTRODUCTION

Hénon-Heiles (HH) systems are Hamiltonian systems in $\mathbb{R}^4$ endowed with the standard symplectic form $dP_1 \wedge dQ_1 + dP_2 \wedge dQ_2$. The Hamiltonian function has the form

$$H = \frac{1}{2}({P_1}^2 + {P_2}^2) + V(Q_1, Q_2)$$

where V is a polynomial function. There are four nontrivial integral cases with quartic potential whose name in the literature is HH4 followed by three numbers giving the ratios of the coefficients of the quartic monomials: 1:2:1, 1:6:1, 1:6:8 and 1:12:16. The generalized HH systems are obtained adding inverse terms to the potential V, without destroying the integrability of the system.

The problem of the integration in quadratures of these systems has been extensively studied in the last decades. The most efficient and elegant method for this purpose, is to find canonical coordinates that separate the Hamilton-Jacobi equation. In this paper we will deal with the delicate task of characterizing such coordinates. The difficulty of the task is well known so that, despite decades of efforts, only one of these four systems has been separated in the generic form: HH4 1:2:1. For the other three systems, the separation coordinates are known only in some degenerate cases. For HH4 1:12:16, the best available results can be found here [7]. In this paper we deal with HH4 1:6:1 and HH4 1:6:8 only.

Let's now introduce them.

2. THE LINK BETWEEN HH4 1:6:1 AND HH4 1:6:8

The generalized Hamiltonian function has the form:

$$H_{161} = \frac{1}{2}({P_1}^2 + {P_2}^2) - \frac{1}{2}\omega\left({Q_1}^2 + {Q_2}^2\right) - \frac{{Q_1}^4}{32} - \frac{3\,{Q_1}^2{Q_2}^2}{16} - \frac{{Q_2}^4}{32} - \frac{{k_1}^2}{2\,{Q_1}^2} - \frac{{k_2}^2}{2\,{Q_2}^2} \tag{1}$$

and depends on three arbitrary constants, ω, k_1 and k_2. The last two terms are the inverse terms and the ratios of the coefficients of the quartic terms are 1:6:1 as expected. This Hamiltonian system possesses an integral of motion that we call K:

$$K_{161} = \left(P_1P_2 - Q_1Q_2\left(\frac{{Q_1}^2}{8} + \frac{{Q_2}^2}{8} + \omega\right)\right)^2 - {k_1}^2\left(\frac{{P_2}^2}{{Q_1}^2} - \frac{{Q_2}^2}{4}\right) - {k_2}^2\left(\frac{{P_1}^2}{{Q_2}^2} - \frac{{Q_1}^2}{4}\right) + \frac{{k_1}^2{k_2}^2}{{Q_1}^2{Q_2}^2} \tag{2}$$

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The reader can easily check that these two functions are in involution with respect to the standard Poisson bracket hence the system is Liouville integrable. The separation coordinates for this system are unknown.

The canonical change of coordinates [1]:

$$\begin{aligned} Q_1 &= \frac{R_+}{2} \\ Q_2 &= \frac{R_-}{2} \\ P_1 &= \frac{R_+}{2}\left(-\frac{p_2}{q_2} - \frac{q_1}{2} - \frac{k_1-k_2}{q_2^2}\right) + 2\frac{k_1}{R_+} \\ P_2 &= \frac{R_-}{2}\left(-\frac{p_2}{q_2} + \frac{q_1}{2} - \frac{k_1-k_2}{q_2^2}\right) - 2\frac{k_2}{R_-} \end{aligned} \tag{3}$$

where

$$R_\pm{}^2 = -4\,q_1{}^2 \pm 8\,p_1 - 2\,q_2{}^2 + 16\,\frac{p_2{}^2}{q_2{}^2} \mp 16\,\frac{q_1 p_2}{q_2} + 32\,\frac{(k_1-k_2)\,p_2}{q_2{}^3} \mp 16\,\frac{(k_1-k_2)\,q_1}{q_2{}^2} + 16\,\frac{(k_1-k_2)^2}{q_2{}^4} + 16\,\omega$$

changes HH4 1:6:1 into HH4 1:6:8:

$$\begin{aligned} h_{168} &= \frac{1}{2}(p_1{}^2 + p_2{}^2) + \frac{\omega\left(4\,q_1{}^2 + q_2{}^2\right)}{2} - \frac{q_1{}^4}{2} - \frac{3\,q_1{}^2 q_2{}^2}{8} - \frac{q_2{}^4}{16} - \gamma\,q_1 + \frac{\beta}{2\,q_2{}^2} \\ k_{168} &= \frac{1}{4}\left(p_2{}^2 - \frac{q_2{}^2\left(2\,q_1{}^2 + q_2{}^2 - 8\,\omega\right)}{8} + \frac{\beta}{q_2{}^2}\right)^2 - \frac{q_2{}^2\left(q_2 p_1 - 2\,q_1 p_2\right)^2}{16} \\ &\quad - \frac{\gamma}{4}\left(2\,\gamma\,q_2{}^2 - 4\,q_2 p_1 p_2 + \frac{q_1 q_2{}^4}{2} + q_1{}^3 q_2{}^2 + 4\,p_2{}^2 q_1 - 4\,\omega q_1 q_2{}^2 + 4\,\frac{q_1\beta}{q_2{}^2}\right). \end{aligned} \tag{4}$$

These functions are usually written in a slightly different form in the literature. It's easy to pass from one form to the other with a simple change of coordinates. The relationships between the coefficients of the two systems are

$$\gamma = \frac{1}{2}(k_1 + k_2) \qquad \beta = -(k_1 - k_2)^2. \tag{5}$$

The separation coordinates of (4), in the case $\gamma = \omega = 0$, were found using Painlevé analysis in 1994 [6]:

$$2q_1{}^2 + q_2{}^2 - \frac{8p_2{}^2 \pm 8\sqrt{R}}{q_2{}^2} \tag{6}$$

where R is the polynomial obtained replacing $\beta = 0$ in k_{168}. The case $\omega \neq 0$ is treated in [8].

Inverting the change of coordinates (3), they provide the separation coordinates for HH4 1:6:1 in the symmetric case $k_1^2 = k_2^2$. As far as we know, no other cases have been separated to this day. In this paper we solve the case $k_1 k_2 = 0$. Before that, let's turn our attention to an alternative method to see the process of separation of coordinates.

3. THE KOWALEWSKI CONDITIONS

In 2005 F. Magri published a paper [3] revisiting the famous problem solved by S. Kowalewski in 1888 [2]: the so called Kowalewski top. The method adopted in the paper is general and can be applied even in the non-Hamiltonian case, provided that a convenient number of commuting vector fields and first integrals are present. It was subsequently refined in several publications over the years and finally presented in a complete form in [4,5], where the reader will find all the proofs that are omitted here.

Let's now summarize the key ideas in the case of a symplectic system in $\mathbb{R}^4$ with Hamiltonian functions H and K.

The method assumes the presence of a second Poisson tensor P_2 compatible with the tensor P_1 associated to the symplectic structure:

$$[P_1, P_2] = [P_2, P_2] = 0$$

where $[\ldots]$ is the Schouten bracket. We also assume that the two Hamiltonian functions H and K are in involution with respect to the Poisson bracket associated to P_2:

$$P_2(dH, dK) = 0.$$

At this stage one can built the torsionless, recursive operator $N = P_2 P_1{}^{-1}$. If N has maximal rank, the two distinct eigenvalues provide (half of) the separation coordinates. The explicit determination of the compatible Poisson tensor P_2, that requires the calculation of six unknown functions, can result quite cumbersome even in relatively simple cases. The number of unknown functions can be reduced to four, the components of a vector field X, looking for tensors $P_2 = L_X(P_1)$. Using this method the bi-Hamiltonian structure of cubic Hénon-Heiles systems can be calculated directly [9].

However, in the present case, the explicit determination of a bi-Hamiltonian structure in natural coordinates seems definitely too complicated. The good news is that one does not have to build up N (and, before this, P_2) in order to calculate its eigenvalues. To understand this point, we start observing that N acts on the vector fields tangent to the Lagrange foliation defined by the level surfaces of H and K [9]. This bi-dimensional foliation is spanned by the Hamiltonian vector fields X_H and X_K so that

$$\begin{aligned} NX_H &= m_1 X_H + m_2 X_K \\ NX_K &= m_3 X_H + m_4 X_K \end{aligned}$$

for some functions $m_1, \dots, m_4$. It is now clear that the so called **control matrix** $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$ is nothing but the restriction of N to the leaves of the foliation, written in the basis associated with X_H and X_K. Furthermore the tensor M is also torsionless since it is the restriction of a torsionless tensor to an invariant surface. This is the first of the two properties that characterize M. The second one is that the vector fields X_H and X_K must commute with respect to the modified commutator

$$[X, Y]_M := [MX, Y] + [X, MY] - M[X, Y]$$

defined on the vector fields tangent to the Lagrangian foliation.

F. Magri proved that these two properties are necessary and sufficient conditions for the system to be separable and for the eigenvalues of M to be separation coordinates [5]. The point of interest in all this discussion is that these two conditions, $T(N) = 0$ and $[X, Y]_M = 0$, can be reduced to four differential constraints, on the entries of M, called Kowalewski Conditions (KC):

$$\begin{aligned} \{m_3, H\} &= \{m_1, K\} \\ \{m_4, H\} &= \{m_2, K\} \\ \{m_1 m_3 + m_3 m_4, H\} &= \{m_1^2 + m_2 m_3, K\} \\ \{m_2 m_3 + m_4^2, H\} &= \{m_1 m_2 + m_2 m_4, K\} \end{aligned} \tag{7}$$

We also need an extra condition for the new coordinates to be canonical: the trace and the determinant of M must be in involution

$$\{m_1 + m_4, m_1 m_4 - m_2 m_3\} = 0. \tag{8}$$

In order to solve the KC one has to solve four differential equations in four unknown functions $m_1, \dots, m_4$; this can be quite challenging. A first step could be to select a particular class of solutions that is easier to calculate but general enough to include most of the significant examples in the literature. The experience suggests this form for M:

$$\begin{aligned} m_1 &= aF^2 + bFG + cG^2 + dF + eG + f \\ m_2 &= gF + v \\ m_3 &= pF^2 + qFG + rG^2 + sF + tG + u \\ m_4 &= gG + w \end{aligned} \tag{9}$$

where all the coefficients $a, b, c\dots$ are constants of the motion while F and G are unknown functions.

If we agree to denote by

$$\dot{f} = X_H(f) \qquad f' = X_K(f)$$

the derivatives of a function f along the given Hamiltonian fields, and replacing (9) into (7), we obtain the following

Proposition 3.1. *If the functions F and G are solutions of the equations*

$$\begin{aligned} F' &= \dot{G} \\ G' &= (\mu F + \tau)\dot{G} - (\mu G + \nu)\dot{F} \end{aligned} \tag{10}$$

where μ, τ and ν are constants of the motion, then the functions

$$\begin{aligned} m_1 &:= -e(\mu F^2 + \tau F - G) - \nu(\mu F + \tau) + w \\ m_2 &:= eF + \nu \\ m_3 &:= -e(\mu FG + \nu F) - \nu(\mu G + \nu) \\ m_4 &:= eG + w \end{aligned} \tag{11}$$

verify the KC.

Therefore, if we limit our search to solutions of the form (9), the problem reduces to two differential equations (10) in two unknown functions F and G.

In [8] we suggested the method of the vector field Z, in order to reduce the task to the search of one single function V (the potential function) and a few constants.

Let's outline the method in the case of HH4 1:6:1. The idea is to extend the phase space including the constants of the problem as new coordinates, so turning our system into a Poisson one. In our example, we can extend the phase space to $\mathbb{R}^6$ with coordinates $(P_1, P_2, Q_1, Q_2, k_1, k_2)$. The Poisson tensor is obtained adding two extra columns and two extra lines of zeros to P_1. We now consider the vector field Z so defined:

$$Z = X_V + w_1 \frac{\partial}{\partial k_1} + w_2 \frac{\partial}{\partial k_2}, \qquad w_1, w_2 \in \mathbb{R}.$$

$V,\ w_1,\ w_2$ are unknown and X_V is the Hamiltonian vector field associated to V. Sometimes it may be useful to look for a potential function of the form $V = \ln f$ (some examples are given in [8]). The next step is to define the "Fundamental Functions" F and G of Proposition 3.1 in this way:

$$F = Z(H_{161}) \qquad G = Z(K_{161}). \tag{12}$$

A simple calculation proves that the first of equations (10) is automatically verified with any choice of the potential function [8]. This means that the problem is finally reduced to the determination of a single function V (plus, eventually, the constants $w_1,\ w_2$) verifying the second equation in (10). It should be stressed that the involutivity condition (8) has to be checked independently from the KC.

It's time now to see how the method of the vector field Z can provide the separation coordinates for HH4 1:6:1 in the case $k_1 k_2 = 0$.

4. HH4 1:6:1 IN THE CASE $k_1 k_2 = 0$

The system (1)-(2) is invariant under the symmetry

$$(P_1, P_2, Q_1, Q_2, k_1, k_2) \longrightarrow (P_2, P_1, Q_2, Q_1, k_2, k_1)$$

so it's enough to solve the case $k_2 = 0$:

$$\begin{aligned} H &= \frac{1}{2}({P_1}^2 + {P_2}^2) - \frac{1}{2}\omega\left({Q_1}^2 + {Q_2}^2\right) - \frac{{Q_1}^4}{32} - \frac{3\,{Q_1}^2{Q_2}^2}{16} - \frac{{Q_2}^4}{32} - \frac{k^2}{2\,{Q_1}^2} \\ K &= \left(P_1P_2 - Q_1Q_2\left(\frac{{Q_1}^2}{8} + \frac{{Q_2}^2}{8} + \omega\right)\right)^2 - k^2\left(\frac{{P_2}^2}{{Q_1}^2} - \frac{{Q_2}^2}{4}\right) \end{aligned} \tag{13}$$

In order to apply the method of the field Z we extend the phase space to $\mathbb{R}^5$ with coordinates (P_1, P_2, Q_1, Q_2, k). A first remark is that the system is homogeneous with respect to the following gradation:

$$P_1,\ P_2,\ \omega \sim 2 \qquad Q_1,\ Q_2 \sim 1 \qquad k \sim 3\,. \tag{14}$$

We detail now the steps of the algorithm.

1. We look for a vector field of the form $Z = X_V + w\,\partial/\partial k$ with $V = \ln f$ as suggested in Section 3. Our problem is now to find the unknown function f and the constant w.
2. Because the system is homogenous we look for homogeneous Fundamental Functions F and G. For that purpose, the presence of the term $\partial/\partial k$ forces $f \sim 4$.
3. Replacing f with the general homogeneous polynomial of degree 4 with respect to the gradation (14) and adding inverse terms like kP_2/Q_1 suggested by the form of the Hamiltonian functions, we are now able to calculate F and G with (12).
4. We can choose the coefficients of f and w in such a way that the second equation in (10) is verified:
$$V = \frac{1}{k}\ln\left(P_1P_2 - Q_1Q_2\left(\frac{{Q_1}^2}{8} + \frac{{Q_2}^2}{8} + \omega\right) + \frac{kP_2}{Q_1} + \frac{kQ_1}{2}\right) \tag{15}$$
and $Z = X_V + \dfrac{1}{k}\dfrac{\partial}{\partial k}$.
5. We choose the integrals of motion in M in such a way that the involutivity condition (8) is verified (see (16)).

The results of this discussion can be summarized in the following

Theorem 4.1. *Consider*

- *the vector field* $Z = X_V + \frac{1}{k}\frac{\partial}{\partial k}$, *where* V *is the function in (15);*
- *the functions* F *and* G *defined in (12);*
- *the Control Matrix*

$$M = \begin{pmatrix} -2k^2F + 2H & 1 \\ -2k^2G + 4K + 8k^2\omega & 2H \end{pmatrix}. \tag{16}$$

Then the eigenvalues of M *are separation (canonical) coordinates for (13).*

Proof. A straightforward calculation gives

$$F = \frac{2\,Q_1{}^2Q_2 + 4\,Q_1P_1 + 4\,k}{Q_1{}^4Q_2 + \left(Q_2{}^3 + 8\,\omega\,Q_2 - 4\,k\right)Q_1{}^2 - 8\,P_1P_2Q_1 - 8\,kP_2}$$

$$G = 2\,\omega - \frac{Q_1{}^2}{4} - \frac{Q_2{}^2}{4} - P_2 - \frac{k\left(Q_1{}^4 + \left(3\,y^2 + 8\,\omega + 4P_2\right)Q_1{}^2 + 4\,Q_1Q_2P_1 + 4\,kQ_2\right)}{Q_1{}^4Q_2 + \left(Q_2{}^3 + 8\,\omega\,Q_2 - 4\,k\right)Q_1{}^2 - 8\,P_1P_2Q_1 - 8\,kP_2}$$

and (16) provides the four entries $m_1, \ldots, m_4$ of the Control Matrix. With these functions (7) and (8) are verified so that the eigenvalues of M are separation coordinates. □

5. FINAL REMARKS

- If we replace K with $K - H^2$ the Control Matrix can be written in the simplified form:

$$M = \begin{pmatrix} -2kF + 4H & 1 \\ -2kG + 4K & 0 \end{pmatrix}.$$

  F. Magri already pointed out a similar behavior of the Kowalewski top [5].
- Another set of separation coordinates can be obtained using quadratic functions in F and G:

$$M = \begin{pmatrix} -2k^2F^2 + G & F \\ -2k^2FG + 4KF & G \end{pmatrix} \tag{17}$$

- HH4 1:6:8 has been solved only in the particular cases $\beta\gamma = 0$ [10]. The eigenvalues of (16) or (17), through the change of coordinates (3), provide the separation coordinates for the case $\beta = -4\gamma^2$.
- HH4 1:6:1 with $k_1 = k_2 = k$ has already been solved using the method of the field Z and a potential function $V = \ln f$ [8]. The functions f for the cases $k_1 = k_2 = k$ and $(k_1, k_2) = (k, 0)$ are, respectively,

$$P_1P_2 - Q_1Q_2\left(\frac{Q_1{}^2}{8} + \frac{Q_2{}^2}{8} + \omega\right) + \frac{kP_2}{Q_1} + \frac{kP_1}{Q_2} + \frac{k^2}{Q_1Q_2}$$

and

$$P_1P_2 - Q_1Q_2\left(\frac{Q_1{}^2}{8} + \frac{Q_2{}^2}{8} + \omega\right) + \frac{kP_2}{Q_1} + \frac{kQ_1}{2}.$$

The idea is to guess, from these particular examples, the form of f for the generic case. The first part of the function is independent from the constants so it is reasonable to expect that it remains unchanged but for the last terms the situation is uncertain. For instance it is not clear why the term $kQ_1/2$ does not appear in the case $k_2 = k_1 = k$.

The generic case remains unsolved.

CONFLICTS OF INTEREST

The author declares no conflicts of interest.

ACKNOWLEDGEMENT

This work was funded by Zayed University, Abu Dhabi, through the RIF Grant R18050.

REFERENCES

[1] S. Baker, V.Z. Enolskii, A.P. Fordy, Integrable quartic potentials and coupled KdV equations, Phys. Lett. A 201 (1995), 167–174.
[2] S. Kowalewski, Sur le problème de la rotation d'un corps solide autour d'un point fixe, Acta Math. 12 (1889), 177–232.
[3] F. Magri, The Kowalevski's top and the method of Syzygies, Ann. Inst. Fourier, Grenoble 55 (2005), 2147–2159.
[4] F. Magri, The Kowalewski's top revisited, 2018. Available from: https://arxiv.org/abs/1809.01879v1.
[5] F. Magri, The Kowalewski separability conditions. in: I. Krichever, S. Novikov, O. Ogievetsky, S. Shlosman (Eds.), Volume Integrability, Quantization, and Geometry. Proceedings of Symposia in Pure Mathematics, AMS, 2021.
[6] V. Ravoson, A. Ramani, B. Grammaticos, Generalized separability for a Hamiltonian with nonseparable quartic potential, Phys. Lett. A 191 (1994), 91–95.
[7] N. Sottocornola, Explicit integration of a generic Hénon-Heiles system with quartic potential, J. Nonlinear Math. Phys. 24 (2017), 346–355.
[8] N. Sottocornola, Separation coordinates in Hénon-Heiles systems, Phys. Lett. A 383 (2019), 126027.
[9] N. Sottocornola, Direct construction of a bi-Hamiltonian structure for cubic Hénon-Heiles systems, J. Geom. Symmetry Phys. 57 (2020), 99–109.
[10] C. Verhoeven, M. Musette, R. Conte, General solution for Hamiltonians with extended cubic and quartic potentials, Theor. Math. Phys. 134 (2003), 128–138.

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics
Vol. **28(3)**; September (2021), *pp.* 309–320
DOI: https://doi.org/10.2991/jnmp.k.210427.001; ISSN 1402-9251; eISSN 1776-0852
https://www.atlantis-press.com/journals/jnmp

Research Article

A Local Equivariant Index Theorem for Sub-Signature Operators

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ARTICLE INFO

Article History

Received 07 February 2021
Accepted 18 April 2021

Keywords

Sub-signature operator
equivariant index

ABSTRACT

In this paper, we prove a local equivariant index theorem for sub-signature operators which generalizes Weiping Zhang's index theorem for sub-signature operators.

1. INTRODUCTION

The Atiyah-Singer index Theorem ([2,3]) gives a cohomological interpretation of the Fredholm index of an elliptic operator. The Atiyah-Bott-Segal-Singer index formula, which called the equivariant index theorem, is a generalization with group action of the Atiyah-Singer index theorem. The first direct proof of this result was given by Patodi, Gilkey, Atiyah-Bott-Patodi partly by using invariant theory [1,12]. This theorem generalizes the Atiyah-Singer index theorem and the Atiyah-Bott fixed point formula for elliptic complexes, which is a generalization of the Lefschetz fixed point formula. In [7], Berline and Vergne gave a heat kernel proof of the Atiyah-Bott-Segal-Singer index formula. Moreover, Lafferty, Yu and Zhang [14] presented a simple and direct geometric proof of the equivariant index theorem for an orientation-preserving isometry on an even dimensional spin manifold by using Clifford asymptotics of heat kernel. Furthermore, Ponge and H. Wang gave a different proof of the equivariant index formula by the Greiner's approach to the heat kernel asymptotics [19]. In [15], in order to prove family rigidity theorems, Liu and Ma proved the equivariant family index formula. In [22], Y. Wang gave another proof of the local equivarint index theorem for a family of Dirac operators by the Greiner's approach to the heat kernel asymptotics. In [21], using the Greiner's approach to the heat kernel asymptotics, Y. Wang proved the equivariant Gauss-Bonnet-Chern formula and gave the variation formulas for the equivariant Ray-Singer metric, which are originally due to J. M. Bismut and W. Zhang [9].

In parallel, Freed [11] considered the case of an orientation reversing involution acting on an odd dimensional spin manifold and gave the associated Lefschetz formulas by the K-theretical way. In [20], Wang constructed an even spectral triple by the Dirac operator and the orientation-reversing involution and computed the Connes-Chern character for this spectral triple. In [16], Liu and Wang proved an equivariant odd index theorem for Dirac operators with involution parity and the Atiyah-Hirzebruch vanishing theorems for odd dimensional spin manifolds. In [24] and [25], Zhang introduced the sub-signature operators and proved a local index formula for these operators. By computing the adiabatic limit of eta-invariants associated to the so-called sub-signature operators, a new proof of the Riemann-Roch-Grothendieck type formula of Bismut-Lott was given in [17] and [10]. The motivation of the present article is to prove a local equivariant index formula for sub-signature operators. As the subsignature operator is locally a twisted Dirac operator, we can obtain our theorem by the proof of equivariant twisted Dirac operators. We give a direct proof of a local equivariant index theorem for subsignature operators by the Volterra calculus, rather than derived from the local equivariant index theorem of twisted Dirac operators. Thus our direct proof of the equivariant index theorem of the subsignature operators using Volterra calculus can be seen as analogous to the works [21,23,26].

This paper is organized as follows: In Section 2, we recall some background on sub-signature operators. In Section 3.1, we prove a local equivariant index formula for sub-signature operators in even dimension. In Section 3.2, we prove a local equivariant odd dimensional index formula for sub-signature operators with an orientation-reversing involution.

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Data availability statement: The authors confirm that the data supporting the findings of this study are available within the article.

2. THE SUB-SIGNATURE OPERATORS

In this section, we give the standard setup (also see Section 1 in [24]). Let M be an oriented closed manifold of dimension n. Let E be an oriented sub-bundle of the tangent vector bundle TM. Let g^{TM} be a metric on TM. Let g^E be the induced metric on E. Let $E^\perp$ be the sub-bundle of TM orthogonal to E with respect to g^{TM}. Let $g^{E^\perp}$ be the metric on $E^\perp$ induced from g^{TM}. Then (TM, g^{TM}) has the following orthogonal splittings

$$TM = E \oplus E^\perp, \tag{2.1}$$

$$g^{TM} = g^E \oplus g^{E^\perp}. \tag{2.2}$$

Clearly, $E^\perp$ carries a canonically induced orientation. We identify the quotient bundle TM/E with $E^\perp$.

Let $\Omega(M) = \bigoplus_0^n \Omega^i(M) = \bigoplus_0^n \Gamma(\wedge^i(T^*M))$ be the set of smooth sections of $\wedge(T^*M)$. Let $*$ be the Hodge star operator of g^{TM}. Then $\Omega(M)$ inherits the following inner product

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \overline{*\beta}, \quad \alpha, \beta \in \Omega(M). \tag{2.3}$$

We use g^{TM} to identify TM and T^*M. For any $e \in \Gamma(TM)$, let $e\wedge$ and i_e be the standard notation for exterior and interior multiplications on $\Omega(M)$. Let $c(e) = e \wedge -i_e$, $\hat{c}(e) = e \wedge +i_e$ be the Clifford actions on $\Omega(M)$ verifying that

$$c(e)c(e') + c(e')c(e) = -2\langle e, e' \rangle_{g^{TM}}, \tag{2.4}$$

$$\hat{c}(e)\hat{c}(e') + \hat{c}(e')\hat{c}(e) = 2\langle e, e' \rangle_{g^{TM}}, \tag{2.5}$$

$$c(e)\hat{c}(e') + \hat{c}(e')c(e) = 0. \tag{2.6}$$

Denote $k = \dim E$ and we assume k is even. Let $\{f_1, \cdots, f_k\}$ be an oriented (local) orthonormal basis of E. Set

$$\hat{c}(E, g^E) = \hat{c}(f_1) \cdots \hat{c}(f_k), \tag{2.7}$$

where $\hat{c}(E, g^E)$ does not depend on the choice of the orthonormal basis. Let

$$\epsilon = \mathrm{Id}_{\wedge^{even}(T^*M)} - \mathrm{Id}_{\wedge^{odd}(T^*M)}$$

be the Z_2-grading operator of

$$\wedge(T^*M) = \wedge^{even}(T^*M) \oplus \wedge^{odd}(T^*M).$$

Set

$$\tau(M, g^E) = \left(\frac{1}{\sqrt{-1}}\right)^{\frac{k(k+1)}{2}} \epsilon \hat{c}(E, g^E). \tag{2.8}$$

It is easy to check

$$\tau(M, g^E)^2 = 1. \tag{2.9}$$

Let

$$\wedge_\pm(T^*M, g^E) = \left\{\omega \in \wedge^*(T^*M), \tau(M, g^E)\omega = \pm\omega\right\}$$

the (even/odd) eigen-bundles of $\tau(M, g^E)$ and by $\Omega_\pm(M, g^E)$ the corresponding set of smooth sections. Let $\delta = \mathrm{d}^*$ be the formal adjoint operator of the exterior differential operator d on $\Omega(M)$ with respect to the inner product (2.3). Set on $\Omega(M) = \Gamma(\Lambda T^*M)$

$$D_E = \frac{1}{2}\left(\hat{c}(E, g^E)(\mathrm{d} + \delta) + (-1)^k(\mathrm{d} + \delta)\hat{c}(E, g^E)\right). \tag{2.10}$$

Then we can check

$$D_E \tau(M, g^E) = -\tau(M, g^E) D_E, \tag{2.11}$$

$$D_E^* = (-1)^{\frac{k(k+1)}{2}} D_E, \tag{2.12}$$

where D_E^* is the formal adjoint operator of D_E with respect to the inner product (2.3). Set

$$\tilde{D}_E = (\sqrt{-1})^{\frac{k(k+1)}{2}} D_E.$$

From (2.11), $\tilde{D}_E$ is a formal self-adjoint first order elliptic differential operator on $\Omega(M)$ interchanging $\Omega_\pm(M, g^E)$.

Definition 2.1. *The sub-signature operator $\tilde{D}_{E,+}$ with respect to (E, g^{TM}) is the restriction of $\tilde{D}_E$ on $\Omega_+(M, g^E)$.*

If we denote the restriction of $\tilde{D}_E$ on $\Omega_\pm(M, g^E)$ by $\tilde{D}_{E,\pm}$, then

$$\tilde{D}^*_{E,\pm} = \tilde{D}_{E,\mp}.$$

Recall that E is the subbundle of TM and that we have the orthogonal decomposition (2.1) of TM and the metric g^{TM}. Let P^E (resp. $P^{E^\perp}$) be the orthogonal projection from TM to E(resp. $E^\perp$). Let ∇^{TM} be the Levi-Civita connection of g^{TM}. We will use the same notation for its lift to $\Omega(M)$. Set

$$\nabla^E = P^E \nabla^{TM} P^E, \tag{2.13}$$

$$\nabla^{E^\perp} = P^{E^\perp} \nabla^{TM} P^{E^\perp}. \tag{2.14}$$

Then ∇^E(resp.$\nabla^{E^\perp}$) is a Euclidean connection on E(resp.$E^\perp$), and we will use the same notation for its lifting on $\Omega(E^*)$(resp. $\Omega(E^{\perp,*})$). Let S be the tensor defined by

$$\nabla^{TM} = \nabla^E + \nabla^{E^\perp} + S.$$

Then S takes values in skew-adjoint endomorphisms of TM, and interchanges E and $E^\perp$. Let $\{e_1, \cdots, e_n\}$ be an oriented (local) orthonormal base of TM. To specify the role of E, set $\{f_1, \cdots, f_k\}$ be an oriented (local) orthonormal basis of E. We will use the greek subscripts for the basis of E. Then by Proposition 1.4 in [24], we have

Proposition 2.2. *The following identity holds,*

$$\tilde{D}_E = (\sqrt{-1})^{\frac{k(k+1)}{2}} \Big(\hat{c}(E, g^E)(\mathrm{d} + \delta) + \frac{1}{2} \sum_i c(e_i)(\nabla^{TM}_{e_i} \hat{c}(E, g^E)) \Big). \tag{2.15}$$

Similar to Lemma 1.1 in [24], we have

Lemma 2.3. *For any $X \in \Gamma(TM)$, the following identity holds,*

$$\nabla^{TM}_X \hat{c}(E, g^E) = -\hat{c}(E, g^E) \sum_\alpha \hat{c}(S(X) f_\alpha) \hat{c}(f_\alpha). \tag{2.16}$$

Let Δ^{TM}, Δ^E be the Bochner Laplacians

$$\Delta^{TM} = \sum_i^n (\nabla^{TM,2}_{e_i} - \nabla^{TM}_{\nabla^{TM}_{e_i} e_i}), \tag{2.17}$$

$$\Delta^E = \sum_i^k (\nabla^{E,2}_{e_i} - \nabla^E_{\nabla^E_{e_i} e_i}). \tag{2.18}$$

Let K be the scalar curvature of (M, g^{TM}). Let R^{TM} (*resp.*, R^E, $R^{E^\perp}$) be the curvature of ∇^{TM} (*resp.*, ∇^E, $\nabla^{E^\perp}$). Let $\{h_1, \cdots, h_{n-k}\}$ be an oriented (local) orthonormal base of $E^\perp$. Now we can state the following Lichnerowicz type formula for $\tilde{D}^2_E$. From Theorem 1.1 in [24], we have

Theorem 2.4. *[24] The following identity holds,*

$$\begin{aligned}
\tilde{D}^2_E = & -\Delta^{TM} + \frac{K}{4} + \frac{1}{8} \sum_{1 \le i,j \le n} \sum_{1 \le \alpha,\beta \le k} \langle R^E(e_i, e_j) f_\beta, f_\alpha \rangle c(e_i) c(e_j) \hat{c}(f_\alpha) \hat{c}(f_\beta) \\
& + \frac{1}{8} \sum_{1 \le i,j \le n} \sum_{1 \le s,t \le n-k} \langle R^{E^\perp}(e_i, e_j) h_t, h_s \rangle c(e_i) c(e_j) \hat{c}(h_s) \hat{c}(h_t) + \frac{1}{2} \sum_\alpha \hat{c} \left((\Delta^{TM} - \Delta^E) f_\alpha \right) \hat{c}(f_\alpha) \\
& + \sum_{i,\alpha} \left(\hat{c}(S(e_i) f_\alpha) \hat{c}(f_\alpha) \nabla^{TM}_{e_i} - \hat{c}(S(e_i) \nabla^E_{e_i} f_\alpha) \hat{c}(f_\alpha) + \frac{1}{2} \hat{c} \left(\nabla^E_{(\nabla^{TM}_{e_i} - \nabla^E_{e_i}) e_i} f_\alpha \right) \hat{c}(f_\alpha) + \frac{3}{4} \parallel S(e_i) f_\alpha) \parallel^2 \right) \\
& + \frac{1}{4} \sum_{i, \alpha \neq \beta} \hat{c}(S(e_i) f_\alpha) \hat{c}(S(e_i) f_\beta) \hat{c}(f_\alpha) \hat{c}(f_\beta).
\end{aligned} \tag{2.19}$$

3. A LOCAL EQUIVARIANT INDEX THEOREM FOR SUB-SIGNATURE OPERATORS

3.1. A Local Even Dimensional Equivariant Index Theorem for Sub-Signature Operators

Let M be a closed oriented Riemannian manifold of even dimension n and ϕ an orientation-preserving isometry on M. Then the smooth map ϕ induces a map $\tilde{\phi} = \phi^{-1,*} : \wedge T^*_x M \to \wedge T^*_{\phi(x)} M$ on the exterior algebra bundle $\wedge T^*_x M$. Let $\tilde{D}_E$ be the sub-signature operator. We assume that $d\phi$ preserves E and $E^{\perp}$ and their orientations, then $\tilde{\phi}\hat{c}(E, g^E) = \hat{c}(E, g^E)\tilde{\phi}$. Then $\tilde{\phi}\tilde{D}_E = \tilde{D}_E\tilde{\phi}$. We will compute the equivariant index

$$\mathrm{Ind}_{\phi}(\tilde{D}_E^+) = \mathrm{Tr}(\tilde{\phi}|_{\ker \tilde{D}_E^+}) - \mathrm{Tr}(\tilde{\phi}|_{\ker \tilde{D}_E^-}). \tag{3.1}$$

We recall the Greiner's approach to the heat kernel asymptotics as in [19] and [4,5,13]. Define the operator given by

$$(Q_0 u)(x,s) = \int_0^{\infty} e^{-s\tilde{D}_E^2}[u(x,t-s)]dt, \quad u \in \Gamma_c(M \times \mathbb{R}, \wedge T^*M), \tag{3.2}$$

maps u continuously to $D'(M \times \mathbb{R}, \wedge T^*M))$ which is the dual space of $\Gamma_c(M \times \mathbb{R}, \wedge T^*M))$. We have

$$\left(\tilde{D}_E^2 + \frac{\partial}{\partial t}\right) Q_0 u = Q_0 \left(\tilde{D}_E^2 + \frac{\partial}{\partial t}\right) u = u, \quad u \in \Gamma_c(M \times \mathbb{R}, \wedge T^*M)). \tag{3.3}$$

Let $(\tilde{D}_E^2 + \frac{\partial}{\partial t})^{-1}$ be the Volterra inverse of $\tilde{D}_E^2 + \frac{\partial}{\partial t}$ as in [5]. That is

$$\left(\tilde{D}_{E,\pm} + \frac{\partial}{\partial t}\right)^{-1}\left(\tilde{D}_{E,\pm} + \frac{\partial}{\partial t}\right) = I - R_1, \quad \left(\tilde{D}_{E,\pm} + \frac{\partial}{\partial t}\right)\left(\tilde{D}_{E,\pm} + \frac{\partial}{\partial t}\right)^{-1} = I - R_2, \tag{3.4}$$

where R_1, R_2 are smoothing operators. Let

$$(Q_0 u)(x,t) = \int_{M \times \mathbb{R}} K_{Q_0}(x,y,t-s)u(y,s)dyds, \tag{3.5}$$

and $k_t(x,y)$ is the heat kernel of $e^{-t\tilde{D}_E^2}$. We get

$$K_{Q_0}(x,y,t) = k_t(x,y) \text{ when } t > 0, \text{ when } t < 0, \ K_{Q_0}(x,y,t) = 0. \tag{3.6}$$

Then Q_0 has the Volterra property, i.e., it has a distribution kernel of the form $K_{Q_0}(x,y,t-s)$ where $K_{Q_0}(x,y,t)$ vanishes on the region $t < 0$. The parabolic homogeneity of the heat operator $\tilde{D}_E^2 + \frac{\partial}{\partial t}$, i.e. the homogeneity with respect to the dilations of $\mathbb{R}^n \times \mathbb{R}^1$ given by

$$\lambda \cdot (\xi, \tau) = (\lambda\xi, \lambda^2\tau), \qquad (\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}^1, \ \lambda \neq 0. \tag{3.7}$$

Let $p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi)$ be the symbol of $\tilde{D}_E^2$, then the symbol of $\tilde{D}_E^2 + \frac{\partial}{\partial t}$ is $\sqrt{-1}\tau + p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi)$, it is homogeneous with respect to (ξ,τ).

In the following, for $g \in S(\mathbb{R}^{n+1})$ and $\lambda \neq 0$, we let g_λ be the tempered distribution defined by

$$\langle g_\lambda(\xi,\tau), u(\xi,\tau)\rangle = |\lambda|^{-(n+2)}\langle g(\xi,\tau), u(\lambda^{-1}\xi, \lambda^{-2}\tau)\rangle, \quad u \in S(\mathbb{R}^{n+1}). \tag{3.8}$$

Definition 3.1. *A distribution $g \in S(\mathbb{R}^{n+1})$ is parabolic homogeneous of degree m, $m \in Z$, if for any $\lambda \neq 0$, we have $g_\lambda = \lambda^m g$.*

Let $\mathbb{C}_-$ denote the complex halfplane $\{\mathrm{Im}\tau < 0\}$ with closure $\overline{\mathbb{C}_-}$. Then:

Lemma 3.2. *[5] Let $q(\xi,\tau) \in C^{\infty}((\mathbb{R}^n \times \mathbb{R})/0)$ be a parabolic homogeneous symbol of degree m such that:*

(i) q extends to a continuous function on $(\mathbb{R}^n \times \overline{\mathbb{C}_-})\backslash 0$ in such way to be holomorphic in the last variable when the latter is restricted to $\mathbb{C}_-$. Then there is a unique $g \in S(\mathbb{R}^{n+1})$ agreeing with q on $\mathbb{R}^{n+1}\backslash 0$ so that:
(ii) g is homogeneous of degree m;
(iii) The inverse Fourier transform $\check{g}(x,t)$ vanishes for $t < 0$.

Let U be an open subset of $\mathbb{R}^n$. We define Volterra symbols and Volterra ΨDOs on $U \times \mathbb{R}^{n+1}\backslash 0$ as follows.

Definition 3.3. *$S_V^m(U \times \mathbb{R}^{n+1})$, $m \in \mathbb{Z}$, consists in smooth functions $q(x,\xi,\tau)$ on $U \times \mathbb{R}^n \times \mathbb{R}$ with an asymptotic expansion $q \sim \sum_{j\geq 0} q_{m-j}$, where:*

(i) $q_l \in C^{\infty}(U \times [(\mathbb{R}^n \times \mathbb{R})/0]$ is a homogeneous Volterra symbol of degree l, i.e. q_l is parabolic homogeneous of degree l and satisfies the property (i) in Lemma 2.3 with respect to the last $n+1$ variables;
(ii) The sign $\sim$ means that, for any integer N and any compact K, U, there is a constant $C_{NK\alpha\beta k} > 0$ such that for $x \in K$ and for $|\xi| + |\tau|^{\frac{1}{2}} > 1$ we have

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta}\partial_{\tau}^{k}(q - \sum_{j<N} q_{m-j})(x,\xi,\tau)| \leq C_{NK\alpha\beta k}(|\xi| + |\tau|^{\frac{1}{2}})^{m-N-|\beta|-2k}. \tag{3.9}$$

Definition 3.4. *$\Psi_V^m(U \times \mathbb{R})$, $m \in \mathbb{Z}$, consists in continuous operators Q_0 from $C_c^{\infty}(U_x \times \mathbb{R}_t)$ to $C^{\infty}(U_x \times \mathbb{R}_t)$ such that:*

(i) Q_0 has the Volterra property;
(ii) $Q_0 = q(x, D_x, D_t) + R$ for some symbol q in $S_V^m(U \times \mathbb{R})$ and some smoothing operator R.

In what follows, if Q_0 is a Volterra ΨDO, we let $K_{Q_0}(x,y,t-s)$ denote its distribution kernel, so that the distribution $K_{Q_0}(x,y,t)$ vanishes for $t<0$.

Definition 3.5. *Let $q_m(x,\xi,\tau)\in C^\infty(U\times(\mathbb{R}^{n+1}/0))$ be a homogeneous Volterra symbol of order m and let $g_m\in C^\infty(U)\otimes\mathbb{S}'(\mathbb{R}^{n+1})$ denote its unique homogeneous extension given by Lemma 2.3. Then:*

(i) $\check{q}_m(x,y,t)$ is the inverse Fourier transform of $g_m(x,\xi,\tau)$ in the last $n+1$ variables;
(ii) $q_m(x,D_x,D_t)$ is the operator with kernel $\check{q}_m(x,y-x,t)$.

Proposition 3.6. *([5,13]) The following properties hold.*

1) *Composition. Let $Q_j\in\Psi_V^{m_j}(U\times\mathbb{R})$, $j=1,2$ have symbol q_j and suppose that Q_1 or Q_2 is properly supported. Then Q_1Q_2 is a Volterra ΨDO of order m_1+m_2 with symbol $q_1\circ q_2\sim\sum\frac{1}{\alpha!}\partial_\xi^\alpha q_1 D_x^\alpha q_2$.*
2) *Parametrices. An operator Q is the order m Volterra ΨDO with the paramatrix P then*

$$QP=1-R_1,\quad PQ=1-R_2 \tag{3.10}$$

where $R_1,\ R_2$ are smoothing operators.

Proposition 3.7. *([5,13]) The differential operator $\tilde{D}_E^2+\partial_t$ is invertible and its inverse $(\tilde{D}_E^2+\partial_t)^{-1}$ is a Volterra ΨDO of order -2.*

We denote by M^ϕ the fixed-point set of ϕ, and for $a=0,\cdots,n$, we let $M^\phi=\bigcup_{0\le a\le n}M_a^\phi$, where M_a^ϕ is an a-dimensional submanifold. Given a fixed-point x_0 in a component M_a^ϕ, consider some local coordinates $x=(x^1,\cdots,x^a)$ around x_0. Setting $b=n-a$, we may further assume that over the range of the domain of the local coordinates there is an orthonormal frame $e_1(x),\cdots,e_b(x)$ of N_z^ϕ. This defines fiber coordinates $v=(v_1,\cdots,v_b)$. Composing with the map $(x,v)\in N^\phi(\varepsilon_0)\to\exp_x(v)$ we then get local coordinates $x^1,\cdots,x^a,v^1,\cdots,v^b$ for M near the fixed point x_0. We shall refer to this type of coordinates as *tubular coordinates.* Then $N^\phi(\varepsilon_0)$ is homeomorphic with a tubular neighborhood of M^ϕ. Set $i_{M^\phi}:M^\phi\hookrightarrow M$ be an inclusion map. Since $d\phi$ preserves E and $E^\perp$, considering the oriented (local) orthonormal basis $\{f_1,\cdots,f_k,h_1,\cdots,h_{n-k}\}$, set

$$d\phi_{x_0}=\begin{pmatrix}\exp(L_1) & 0\\ 0 & \exp(L_2)\end{pmatrix}, \tag{3.11}$$

where $L_1\in\mathfrak{so}(k)$ and $L_2\in\mathfrak{so}(n-k)$

Let

$$\widehat{A}(R^{M^\phi})=\det{}^{\frac12}\left(\frac{R^{M^\phi}/4\pi}{\sinh(R^{M^\phi}/4\pi)}\right);\ \ \nu_\phi(R^{N^\phi}):=\det{}^{-\frac12}(1-\phi^Ne^{-\frac{R^{N^\phi}}{2\pi}}). \tag{3.12}$$

The aim of this section is to prove the following result.

Theorem 3.8. *(Local Equivariant Sub-Signature Index Theorem. Even Dimension)*

Let $x_0\in M^\phi$, then

$$\begin{aligned}\lim_{t\to0}\mathrm{Str}\left[\tilde{\phi}(x_0)K_t(x_0,\phi(x_0))\right]=&\left(\frac{1}{\sqrt{-1}}\right)^{\frac{k}{2}}2^{\frac{n}{2}}\left\{\widehat{A}(R^{M^\phi})\nu_\phi(R^{N^\phi})i^*_{M^\phi}\left[\det{}^{\frac12}\left(\cosh\left(\frac{R^E}{4\pi}-\frac{L_1}{2}\right)\right)\right.\right.\\ &\left.\left.\times\det{}^{\frac12}\left(\frac{\sinh\left(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\right)}{\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}}\right)\mathrm{Pf}\left(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\right)\right]\right\}^{(a,0)}(x_0),\end{aligned} \tag{3.13}$$

where $L_1\in so(k),L_2\in so(n-k)$ and $\mathrm{Pf}\left(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\right)$ *denotes the Pfaffian of* $\left(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\right)$.

Next we give a detailed proof of Theorem 3.9. Let $Q=(\tilde{D}_E^2+\partial_t)^{-1}$. For $x\in M^\phi$ and $t>0$ set

$$I_Q(x,t):=\widetilde{\phi}(x)^{-1}\int_{N_x^\phi(\varepsilon)}\phi(\exp_xv)K_Q(\exp_xv,\exp_x(\phi'(x)v),t)dv. \tag{3.14}$$

Here we use a trivialization over $\wedge(T^*M)$ about the tubular coordinates. Using the tubular coordinates, we have

$$I_Q(x,t)=\int_{|v|<\varepsilon}\widetilde{\phi}(x,0)^{-1}\widetilde{\phi}(x,v)K_Q(x,v;x,\phi'(x)v;t)dv. \tag{3.15}$$

Let

$$q_{m-j}^{\wedge(T^*M)}(x,v;\xi,v;\tau) := \widetilde{\phi}(x,0)^{-1}\widetilde{\phi}(x,v)q_{m-j}(x,v;\xi,v;\tau). \tag{3.16}$$

We mention the following result

Proposition 3.9. *[19] Let $Q \in \Psi_V^m(M\times\mathbb{R},\wedge(T^*M))$, $m\in\mathbb{Z}$. Uniformly on each component M_a^ϕ*

$$I_Q(x,t) \sim \sum_{j\geq 0} t^{-(\frac{a}{2}+[\frac{m}{2}]+1)} I_Q^j(x) \quad \text{as } t\to 0^+, \tag{3.17}$$

where $I_Q^j(x)$ is defined by

$$I_Q^{(j)}(x) := \sum_{|\alpha|\leq m-[\frac{m}{2}]+2j} \int \frac{v^\alpha}{\alpha!}\left(\partial_v^\alpha q_{2[\frac{m}{2}]-2j+|\alpha|}^{\wedge(T^*M)}\right)^\vee (x,0;0,(1-\phi'(x))v;1)dv. \tag{3.18}$$

Similar to Theorem 1.2 in [15] and Section 2 (d) in [8], we have

$$\begin{aligned}\mathrm{Str}_\tau[\tilde{\phi}\exp(-t\tilde{D}_E^2)] &= (\sqrt{-1})^{\frac{k}{2}}\int_M \mathrm{Str}_\epsilon\left[\hat{c}(E,g^E)k_t(x,\phi(x))\right]\mathrm{d}x\\ &= (\sqrt{-1})^{\frac{k}{2}}\int_M \mathrm{Str}_\epsilon[\hat{c}(E,g^E)K_{(\tilde{D}_E^2+\partial_t)^{-1}}(x,\phi(x),t)]\mathrm{d}x.\end{aligned} \tag{3.19}$$

We will compute the local index in this trivialization. Let (V,q) be a finite dimensional real vector space equipped with a quadratic form. Let $C(V,q)$ be the associated Clifford algebra, i.e., the associative algebra generated by V with the relations $v\cdot w+w\cdot v=-2q(v,w)$ for $v,w\in V$. Let $\{e_1,\cdots,e_n\}$ be an orthomormal basis of (V,q), let $C(V,q)\hat{\otimes}C(V,-q)$ be the grading tensor product of $C(V,q)$ and $C(V,-q)$, and $\wedge^*V\hat{\otimes}\wedge^*V$ be the grading tensor product of $\wedge^*V$ and $\wedge^*V$. Define the symbol map:

$$\sigma: C(V,q)\hat{\otimes}C(V,-q)\to\wedge^*V\hat{\otimes}\wedge^*V; \tag{3.20}$$

where $\sigma(c(e_{j_1})\cdots c(e_{j_l})\otimes 1)=e^{j_1}\wedge\cdots\wedge e^{j_1}\otimes 1$, $\sigma(1\otimes\hat{c}(e_{j_1})\cdots\hat{c}(e_{j_l}))=1\otimes\hat{e}^{j_1}\wedge\cdots\wedge\hat{e}^{j_1}$. Using the interior multiplication $\iota(e_j):\wedge^*V\to\wedge^{*-1}V$ and the exterior multiplication $\varepsilon(e_j):\wedge^*V\to\wedge^{*+1}V$, we define representations of $C(V,q)$ and $C(V,-q)$ on the exterior algebra:

$$c: C(V,q)\to\mathrm{End}\wedge V,\ e_j\mapsto c(e_j):\ \varepsilon(e_j)-\iota(e_j); \tag{3.21}$$

$$\hat{c}: C(V,-q)\to\mathrm{End}\wedge V,\ e_j\mapsto\hat{c}(e_j):\ \varepsilon(e_j)+\iota(e_j). \tag{3.22}$$

The tensor product of these representations yields an isomorphism of superalgebras

$$c\otimes\hat{c}: C(V,q)\hat{\otimes}C(V,-q)\to\mathrm{End}\wedge V \tag{3.23}$$

which we will also denote by c. We obtain a supertrace (i.e., a linear functional vanishing on supercommutators) on $C(V,q)\hat{\otimes}C(V,-q)$ by setting $\mathrm{Str}(a)=\mathrm{Str}_{\mathrm{End}\wedge V}[c(a)]$ for $a\in C(V,q)\hat{\otimes}(V,-q)$, where $\mathrm{Str}_{\mathrm{End}\wedge V}$ is the canonical supertrace on $\mathrm{End}V$.

Lemma 3.10. *For $1\leq i_1<\cdots<i_p\leq n$, $1\leq j_1<\cdots<j_q\leq n$, when $p=q=n$,*

$$\mathrm{Str}[c(e_{i_1})\cdots c(e_{i_n})\hat{c}(e_{i_1})\cdots\hat{c}(e_{i_n})]=(-1)^{\frac{n(n+1)}{2}}2^n \tag{3.24}$$

and otherwise equals zero.

We will also denote the volume element in $\wedge V\hat{\otimes}\wedge V$ by $\omega=e^1\wedge\cdots\wedge e^n\wedge\hat{e}^1\wedge\cdots\wedge\hat{e}^n$. For $a\in\wedge V\hat{\otimes}\wedge V$, let $\mathrm{T}a$ be the coefficient of ω. The linear functional $\mathrm{T}:\wedge V\hat{\otimes}\wedge V\to\mathrm{R}$ is called the Berezin trace. Then for a $a\in C(V,q)\hat{\otimes}(V,.q)$, we have $\mathrm{Str}_s(a)=(-1)^{\frac{n(n+1)}{2}}2^n(\mathrm{T}\sigma)(a)$. We define the Getzler order as follows:

$$\deg\partial_j=\frac{1}{2}\deg\partial_t=-\deg x^j=1,\quad \deg c(e_j)=1,\ \deg\hat{c}(e_j)=0. \tag{3.25}$$

Let $Q\in\Psi_V^*(\mathbb{R}^n\times\mathbb{R},\wedge^*T^*M)$ have symbol

$$q(x,\xi,\tau)\sim\sum_{k\leq m'}q_k(x,\xi,\tau), \tag{3.26}$$

where $q_k(x,\xi,\tau)$ is an order k symbol. Then taking components in each subspace $\wedge^jT^*M\otimes\wedge^lT^*M$ of $\wedge T^*M\otimes\wedge T^*M$ and using Taylor expansions at $x=0$ give formal expansions

$$\sigma[q(x,\xi,\tau)]\sim\sum_{j,k}\sigma[q_k(x,\xi,\tau)]^{(j,l)}\sim\sum_{j,k,\alpha}\frac{x^\alpha}{\alpha!}\sigma[\partial_x^\alpha q_k(0,\xi,\tau)]^{(j,l)}. \tag{3.27}$$

The symbol $\frac{x^\alpha}{\alpha!}\sigma[\partial_x^\alpha q_k(0,\xi,\tau)]^{(j,l)}$ is the Getzler homogeneous of $k+j-|\alpha|$. Therefore, we can expand $\sigma[q(x,\xi,\tau)]$ as

$$\sigma[q(x,\xi,\tau)] \sim \sum_{j\geq 0} q_{(m-j)}(x,\xi,\tau), \qquad q_{(m)} \neq 0, \tag{3.28}$$

where $q_{(m-j)}$ is a Getzler homogeneous symbol of degree $m-j$.

Definition 3.11. *The integer m is called as the Getzler order of Q. The symbol $q_{(m)}$ is the principal Getzler homogeneous symbol of Q. The operator $Q_{(m)} = q_{(m)}(x, D_x, D_t)$ is called the model operator of Q.*

Let $e_1,\ldots,e_n$ be an oriented orthonormal basis of $T_{x_0}M$ such that $e_1,\cdots,e_a$ span $T_{x_0}M^\phi$ and $e_{a+1},\cdots,e_n$ span $N^\phi_{x_0}$. This provides us with normal coordinates $(x_1,\cdots,x_n)\to \exp_{x_0}(x^1e_1+\cdots+x^ne_n)$. Moreover using parallel translation enables us to construct a synchronous local oriented tangent frame $e_1(x),...,e_n(x)$ such that $e_1(x),\cdots,e_a(x)$ form an oriented frame of TM^ϕ_a and $e_{a+1}(x),\cdots,e_n(x)$ form an (oriented) frame N^τ (when both frames are restricted to M^ϕ). This gives rise to trivializations of the tangent and exterior algebra bundles. Write

$$\phi'(0) = \begin{pmatrix} 1 & 0 \\ 0 & \phi^N \end{pmatrix} = \exp(A_{ij}), \tag{3.29}$$

where $A_{ij}\in \mathfrak{so}(n)$.

Let $\wedge(n) = \wedge^*\mathbb{R}^n$ be the exterior algebra of $\mathbb{R}^n$. We shall use the following gradings on $\wedge(n)\hat{\otimes}\wedge(n)$,

$$\wedge(n)\hat{\otimes}\wedge(n) = \bigoplus_{\substack{1\leq k_1,k_2\leq a \\ 1\leq \bar{l}_1,\bar{l}_2\leq b}} \wedge^{k_1,\bar{l}_1}(n)\hat{\otimes}\wedge^{k_2,\bar{l}_2}(n), \tag{3.30}$$

where $\wedge^{k,\bar{l}}(n)$ is the space of forms $dx^{i_1}\wedge\cdots\wedge dx^{i_{k+\bar{l}}}$ with $1\leq i_1<\cdots<i_k\leq a$ and $a+1\leq i_{k+1}<\cdots<i_{k+\bar{l}}\leq n$. Given a form $\omega\in\wedge(n)\hat{\otimes}\wedge(n)$, denote by $\omega^{(k_1,\bar{l}_1),(k_2,\bar{l}_2)}$ its component in $\wedge(n)^{(k_1,\bar{l}_1)}\hat{\otimes}\wedge^{(k_2,\bar{l}_2)}(n)$. We denote by $|\omega|^{(a,0),(a,0)}$ the Berezin integral $|\omega^{(*,0),(*,0)}|^{(a,0),(a,0)}$ of its component $\omega^{(*,0),(*,0)}$ in $\wedge^{(*,0),(*,0)}(n)$.

Let $A\in Cl(V,q)\hat{\otimes}Cl(V,-q)$, then

$$\begin{aligned}\mathrm{Str}[\widetilde{\phi}A] &= (-1)^{\frac{n}{2}}2^n(-\frac{1}{4})^{\frac{b}{2}}det(1-\phi^N)|\sigma(A)|^{((a,0),(a,0))} \\ &+(-1)^{\frac{n}{2}}2^n\sum_{0\leq l_1<b,0\leq l_2\leq b}|\sigma(\widetilde{\phi})^{((0,l_1),(0,l_2))}\sigma(A)^{((a,b-l_1),(a,b-l_2))}|^{(n,n)}.\end{aligned} \tag{3.31}$$

In order to calculate $\mathrm{Str}[\widetilde{\phi}A]$, we need to consider the representation of $|\sigma(\widetilde{\phi})^{((0,b),(0,l_2))}\sigma(A)^{((a,0),(a,b-l_2))}|^{(n,n)}$. Let the matrix ϕ^N equal

$$\phi^N = \begin{pmatrix} A_{\frac{a}{2}+1} & & & \\ & \ddots & 0 & \\ & & \ddots & \\ & 0 & & \ddots \\ & & & A_{\frac{n}{2}} \end{pmatrix}, \ A_{\frac{a}{2}+1} = \begin{pmatrix} \cos\theta_{\frac{a}{2}+1} & \sin\theta_{\frac{a}{2}+1} \\ -\sin\theta_{\frac{a}{2}+1} & \cos\theta_{\frac{a}{2}+1} \end{pmatrix}, \ A_{\frac{n}{2}} = \begin{pmatrix} \cos\theta_{\frac{n}{2}} & \sin\theta_{\frac{n}{2}} \\ -\sin\theta_{\frac{n}{2}} & \cos\theta_{\frac{n}{2}} \end{pmatrix}. \tag{3.32}$$

From Lemma 3.2 in [26], then

Lemma 3.12. *We have*

$$\begin{aligned}\tilde{\phi} = \left(\frac{1}{2}\right)^{\frac{n-a}{2}}\prod_{j=\frac{a}{2}+1}^{n}&\Big[(1+\cos\theta_j)-(1-\cos\theta_j)c(e_{2j-1})c(e_{2j})\hat{c}(e_{2j-1})\hat{c}(e_{2j}) \\ &+\sin\theta_j\left(c(e_{2j-1})c(e_{2j})-\hat{c}(e_{2j-1})\hat{c}(e_{2j})\right)\Big].\end{aligned} \tag{3.33}$$

Then we obtain

$$\begin{aligned}\sigma(\tilde{\phi})^{((0,b),(0,l_2))} &= \left(\frac{1}{2}\right)^{\frac{n-a}{2}}\sigma\left\{\prod_{j=\frac{a}{2}+1}^{n}\left[-(1-\cos\theta_j)c(e_{2j-1})c(e_{2j})\hat{c}(e_{2j-1})\hat{c}(e_{2j})+\sin\theta_j\left(c(e_{2j-1})c(e_{2j})\right)\right]\right\}^{((0,b),(0,l_2))} \\ &= \left(\frac{1}{2}\right)^{\frac{n-a}{2}}e^{a+1}\wedge\cdots\wedge e^n\sigma\left\{\prod_{j=\frac{a}{2}+1}^{n}\left[-(1-\cos\theta_j)\hat{c}(e_{2j-1})\hat{c}(e_{2j})+\sin\theta_j\right]\right\}^{(0,l_2)}\end{aligned}$$

$$= \left(\frac{1}{2}\right)^{\frac{n-a}{2}} e^{a+1}\wedge\cdots\wedge e^n\sigma\left\{\prod_{j=\frac{a}{2}+1}^{n} 2\sin\frac{\theta_j}{2}\left[\cos\frac{\theta_j}{2}-\sin\frac{\theta_j}{2}\hat{c}(e_{2j-1})\hat{c}(e_{2j})\right]\right\}^{(0,l_2)}$$

$$= \left(\frac{1}{2}\right)^{\frac{n-a}{2}} e^{a+1}\wedge\cdots\wedge e^n\det{}^{\frac{1}{2}}(1-\phi^N)\sigma\left[\exp\left(-\frac{1}{4}\sum_{1\le i,j\le n}A_{ij}\hat{c}(e_i)\hat{c}(e_j)\right)\right]^{(0,l_2)}$$

$$= \left(\frac{1}{2}\right)^{\frac{n-a}{2}} e^{a+1}\wedge\cdots\wedge e^n\det{}^{\frac{1}{2}}(1-\phi^N)\sigma\left[\exp\left(-\frac{1}{4}\sum_{1\le i,j\le k}(L_1)_{ij}\hat{c}(f_i)\hat{c}(f_j)\right.\right.$$

$$\left.\left.-\frac{1}{4}\sum_{1\le i,j\le n-k}(L_2)_{k+i,k+j}\hat{c}(h_i)\hat{c}(h_j)\right)\right]^{(0,l_2)}. \quad (3.34)$$

Next we calculate $|\sigma(A)|^{((a,0),(a,b-l_2))}$. In the following, we shall use the following "curvature forms": $R' := (R_{i,j})_{1\le i,j\le a}$, $R'' := (R_{a+i,a+j})_{1\le i,j\le b}$. Let

$$\dot{R} = \frac{1}{4}\sum_{1\le\alpha,\beta\le k}\langle R^E f_\alpha, f_\beta\rangle\hat{c}(f_\alpha)\hat{c}(f_\beta),$$

$$\ddot{R} = \frac{1}{4}\sum_{1\le s,t\le n-k}\langle R^{E^\perp} h_s, h_t\rangle\hat{c}(h_s)\hat{c}(h_t);$$

and

$$\tilde{\dot{R}} = \frac{1}{4}\sum_{1\le\alpha,\beta\le k}\langle (R^E-L_1)f_\alpha, f_\beta\rangle\hat{c}(f_\alpha)\hat{c}(f_\beta),$$

$$\tilde{\ddot{R}} = \frac{1}{4}\sum_{1\le s,t\le n-k}\langle (R^{E^\perp}-L_2)h_s, h_t\rangle\hat{c}(h_s)\hat{c}(h_t).$$

By (2.19), let $F=\tilde{D}_E^2$, we get

Proposition 3.13. *The model operator of F is*

$$F_{(2)} = -\sum_{r=1}^{n}\left(\partial_r+\frac{1}{8}\sum_{1\le i,j,l\le n}\langle R^{TM}(e_i,e_j)e_l,e_r\rangle y_l e^i\wedge e^j\right)^2+\frac{1}{8}\sum_{1\le i,j\le n}\sum_{1\le\alpha,\beta\le k}\langle R^E(e_i,e_j)f_\beta,f_\alpha\rangle e^i\wedge e^j\hat{c}(f_\alpha)\hat{c}(f_\beta)$$
$$+\frac{1}{8}\sum_{1\le i,j\le n}\sum_{1\le s,t\le n-k}\langle R^{E^\perp}(e_i,e_j)h_t,h_s\rangle e^i\wedge e^j\hat{c}(h_s)\hat{c}(h_t). \quad (3.35)$$

From the representation of $F_{(2)}$, we get the model operator of $\frac{\partial}{\partial t}+\tilde{D}_E^2$ is $\frac{\partial}{\partial t}+F_{(2)}$. And we have

$$\left(\frac{\partial}{\partial t}+F_{(2)}\right)K_{Q_{(-2)}}(x,y,t)=0. \quad (3.36)$$

Similar to Lemma 2.9 in [19], we get

Lemma 3.14. *Let $Q\in\Psi^{(-2)}(\mathbb{R}^n\times\mathbb{R},\wedge(T^*M))$ be a parametrix for $(F_{(2)}+\partial_t)^{-1}$. Then*
(1) Q has Getzler order -2 and its model operator is $(F_{(2)}+\partial_t)^{-1}$.
(2) For all $t>0$,

$$(\sqrt{-1})^{\frac{k}{2}}\hat{c}(E,g^E)I_{(F_{(2)}+\partial_t)^{-1}}(0,t)=(\sqrt{-1})^{\frac{k}{2}}\hat{c}(E,g^E)\frac{(4\pi t)^{-\frac{a}{2}}}{\det^{\frac{1}{2}}(1-\phi^N)}\det{}^{\frac{1}{2}}\left(\frac{\frac{tR'}{2}}{\sinh\left(\frac{tR'}{2}\right)}\right)\det{}^{-\frac{1}{2}}(1-\phi^N e^{-tR''})\exp\left(t(\tilde{\dot{R}}+\tilde{\ddot{R}})\right). \quad (3.37)$$

Similar to Lemma 3.6 in [22]. we have

Lemma 3.15. *$Q\in\Psi_V^*(\mathbb{R}^n\times\mathbb{R},\wedge(T^*M))$ has the Getzler order m and model operator $Q_{(m)}$. Then as $t\to 0^+$*

(1) $\sigma[I_Q(0,t)]^{(j,l)}=O(t^{\frac{j-m-a-1}{2}})$, if $m-j$ is odd.
(2) $\sigma[I_Q(0,t)]^{(j,l)}=O(t^{\frac{j-m-a-2}{2}})I_{Q(m)}(0,1)^{(j,l)}+O(t^{\frac{j-m-a}{2}})$, if $m-j$ is even.

In particular, for $m=-2$ and $j=a$ and a is even we get

$$\sigma[I_Q(0,t)]^{((a,0),(a,b-l_2))} = I_{Q(-2)}(0,1)^{((a,0),(a,b-l_2))} + O(t^{\frac{1}{2}}). \tag{3.38}$$

With all these preparations, we are going to prove the local even dimensional equivariant index theorem for sub-signature operators. Substituting (3.34), (3.37) into (3.31), we obtain

$$\begin{aligned}
&\lim_{t\to 0}\mathrm{Str}_\varepsilon\left[\tilde{\phi}(x_0)(\sqrt{-1})^{\frac{k}{2}}\hat{c}(E,g^E)I_{(F+\partial_t)^{-1}}(x_0,t)\right]\\
&= (-1)^{\frac{n}{2}}2^n\left(\frac{1}{2}\right)^{\frac{n-a}{2}}(4\pi)^{-\frac{a}{2}}(\sqrt{-1})^{\frac{k}{2}}\left|\widehat{A}(R^{M^\phi})\nu_\phi(R^{N^\phi})\sigma\left[\hat{c}(f_1)\cdots\hat{c}(f_k)\exp(\tilde{\tilde{R}}+\tilde{\tilde{\tilde{R}}})\right]\right|^{((a,0),n)}\\
&= \left(\frac{1}{\sqrt{-1}}\right)^{\frac{k}{2}}2^{\frac{n}{2}}\left\{\widehat{A}(R^{M^\phi})\nu_\phi(R^{N^\phi})i^*_{M^\phi}\left[\det{}^{\frac{1}{2}}\left(\cosh\left(\frac{R^E}{4\pi}-\frac{L_1}{2}\right)\right)\right.\right.\\
&\quad\left.\left.\times\det{}^{\frac{1}{2}}\left(\frac{\sinh\left(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\right)}{\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}}\right)\mathrm{Pf}\left(\frac{R^{E^\perp}}{4\pi}-\frac{L_2}{2}\right)\right]\right\}^{(a,0)}(x_0).
\end{aligned} \tag{3.39}$$

Where we have used the algebraic result of Proposition 3.13 in [6], and the Berezin integral in the right hand side of (3.39) is the application of the following lemma.

Lemma 3.16. *Let $L_1\in so(k), L_2\in so(n-k)$, we have*

$$\begin{aligned}
\left|\sigma\left[\hat{c}(f_1)\cdots\hat{c}(f_k)\exp(\tilde{\tilde{R}}+\tilde{\tilde{\tilde{R}}})\right]\right|^{(n)} &= (-1)^{\frac{n-k}{2}}\det{}^{\frac{1}{2}}\left(\cosh\left(\frac{R^E-L_1}{2}\right)\right)\\
&\quad\times\det{}^{\frac{1}{2}}\left(\frac{\sinh\left(\frac{R^{E^\perp}-L_2}{2}\right)}{(R^{E^\perp}-L_2)/2}\right)\mathrm{Pf}\left(\frac{R^{E^\perp}-L_2}{2}\right).
\end{aligned} \tag{3.40}$$

Proof. In order to compute this differential form, we make use of the Chern root algorithm (see [22]). Assume that $n=\dim M$ and $k=\dim E$ are both even integers. As in [7], let $L_1\in so(k), L_2\in so(n-k)$, we write

$$R^E-L_1=\begin{pmatrix}\begin{pmatrix}0&-\theta_1\\\theta_1&0\end{pmatrix}&&0\\&\ddots&\\0&&\begin{pmatrix}0&-\theta_{-\frac{k}{2}}\\\theta_{-\frac{k}{2}}&0\end{pmatrix}\end{pmatrix},\ R^{E^\perp}-L_2=\begin{pmatrix}\begin{pmatrix}0&-\hat{\theta}_1\\\hat{\theta}_1&0\end{pmatrix}&&0\\&\ddots&\\0&&\begin{pmatrix}0&-\hat{\theta}_{\frac{n-k}{2}}\\\hat{\theta}_{\frac{n-k}{2}}&0\end{pmatrix}\end{pmatrix}. \tag{3.41}$$

Then we obtain

$$\begin{aligned}
\frac{1}{4}\sum_{1\le\alpha,\beta\le k}\langle(R^E-L_1)f_\alpha,f_\beta\rangle\hat{c}(f_\alpha)\hat{c}(f_\beta) &= \frac{1}{2}\sum_{1\le\alpha<\beta\le k}\langle(R^E-L_1)f_\alpha,f_\beta\rangle\hat{c}(f_\alpha)\hat{c}(f_\beta)\\
&= \frac{1}{2}\sum_{1\le j\le\frac{k}{2}}\theta_j\hat{c}(f_{2j-1})\hat{c}(f_{2j});
\end{aligned} \tag{3.42}$$

$$\begin{aligned}
\frac{1}{4}\sum_{1\le s,t\le n-k}\langle(R^{E^\perp}-L_2)h_s,h_t\rangle\hat{c}(h_s)\hat{c}(h_t) &= \frac{1}{2}\sum_{1\le s<t\le n-k}\langle(R^{E^\perp}-L_2)h_s,h_t\rangle\hat{c}(h_s)\hat{c}(h_t)\\
&= \frac{1}{2}\sum_{1\le l\le\frac{n-k}{2}}\hat{\theta}_l\hat{c}(h_{2l-1})\hat{c}(h_{2l}).
\end{aligned} \tag{3.43}$$

Then the left hand side of (3.40) is

$$\begin{aligned}
&\left|\sigma\left(\hat{c}(f_1)\cdots\hat{c}(f_k)\exp(\tilde{\tilde{R}}+\tilde{\tilde{\tilde{R}}})\right)\right|^{(n)}\\
&=\left|\sigma\left(\hat{c}(f_1)\cdots\hat{c}(f_k)\prod_{1\le j\le\frac{k}{2}}\exp\left(\frac{1}{2}\theta_j\hat{c}(f_{2j-1})\hat{c}(f_{2j})\right)\prod_{1\le l\le\frac{n-k}{2}}\exp\left(\frac{1}{2}\hat{\theta}_l\hat{c}(h_{2l-1})\hat{c}(h_{2l})\right)\right)\right|^{(n)}
\end{aligned}$$

$$= \left|\sigma\left(\hat{c}(f_1)\cdots\hat{c}(f_k)\prod_{1\le j\le \frac{k}{2}}\left[\cos\frac{\theta_j}{2}-\sin\frac{\theta_j}{2}\hat{c}(f_{2j-1})\hat{c}(f_{2j})\right]\prod_{1\le l\le \frac{n-k}{2}}\left[\cos\frac{\hat{\theta}_l}{2}-\sin\frac{\hat{\theta}_l}{2}\hat{c}(h_{2l-1})\hat{c}(h_{2l})\right]\right)\right|^{(n)}$$

$$= (-1)^{\frac{n-k}{2}}\prod_{1\le j\le \frac{k}{2}}\cos\frac{\theta_j}{2}\prod_{1\le l\le \frac{n-k}{2}}\sin\frac{\hat{\theta}_l}{2}. \tag{3.44}$$

Now we consider the right hand side of (3.40),

$$(R^E-L_1)^{2p} = (-1)^p\begin{pmatrix}\begin{pmatrix}\theta_1^{2p} & 0\\ 0 & \theta_1^{2p}\end{pmatrix} & & 0\\ & \ddots & \\ 0 & & \begin{pmatrix}\theta_{\frac{k}{2}}^{2p} & 0\\ 0 & \theta_{\frac{k}{2}}^{2p}\end{pmatrix}\end{pmatrix}. \tag{3.45}$$

Then

$$\det{}^{\frac{1}{2}}\left(\cosh\left(\frac{R^E-L_1}{2}\right)\right) = \prod_{j=1}^{\frac{k}{2}}\left(\sum_{p=0}^{\infty}\left(\frac{\theta_j}{2}\right)^{2p}\frac{(-1)^p}{(2p)!}\right) = \prod_{j=1}^{\frac{k}{2}}\cosh\frac{\sqrt{-1}\theta_j}{2} = \prod_{j=1}^{\frac{k}{2}}\frac{e^{\frac{\sqrt{-1}\theta_j}{2}}+e^{-\frac{\sqrt{-1}\theta_j}{2}}}{2} = \prod_{j=1}^{\frac{k}{2}}\cos\frac{\theta_j}{2}. \tag{3.46}$$

Similarly, we have

$$\det{}^{\frac{1}{2}}\left(\frac{\sinh\left(\frac{R^{E^\perp}-L_2}{2}\right)}{(R^{E^\perp}-L_2)/2}\right) = \prod_{j=1}^{\frac{n-k}{2}}\frac{\sin\frac{\hat{\theta}_j}{2}}{\frac{\hat{\theta}_j}{2}}. \tag{3.47}$$

On the other hand,

$$\mathrm{Pf}\left(\frac{R^{E^\perp}-L_2}{2}\right) = T\left(\exp\left(\sum_{s<t}\langle\frac{R^{E^\perp}-L_2}{2}h_s,h_t\rangle h^s\wedge h^t\right)\right) = T\left(\exp\left(\sum_{1\le j\le\frac{n-k}{2}}\frac{\hat{\theta}_j}{2}h^{2j-1}\wedge h^{2j}\right)\right) = \prod_{j=1}^{\frac{n-k}{2}}\frac{\hat{\theta}_j}{2}. \tag{3.48}$$

Combining these equations, the proof of lemma 3.17 is complete. □

To summarize, we have proved Theorem 3.9.

3.2. The Local Odd Dimensional Equivariant Index Theorem for Sub-Signature Operators

In this section, we give a proof of a local odd dimensional equivariant index theorem for sub-signature operators. Let M be an odd dimensional oriented closed Riemannian manifold. Using (2.19) in Section 2, we may define the sub-signature operators $\tilde{D}_E$. Let γ be an orientation reversing involution isometric acting on M. Let $d\gamma$ preserve $E, E^\perp$ and preserve the orientation of E, then $\tilde{\gamma}\hat{\tau}(E,g^E) = \hat{\tau}(E,g^E)\tilde{\gamma}$, where $\tilde{\gamma}$ is the lift on the exterior algebra bundle $\wedge T^*M$ of $d\gamma$. There exists a self-adjoint lift $\widetilde{\gamma}: \Gamma(M;\wedge(T^*M))\to\Gamma(M;\wedge(T^*M))$ of $d\gamma$ satisfying

$$\widetilde{\gamma}^2 = 1;\ \ \tilde{D}_E\widetilde{\gamma} = -\widetilde{\gamma}\tilde{D}_E. \tag{3.49}$$

Now the $+1$ and -1 eigenspaces of $\widetilde{\gamma}$ give a splitting

$$\Gamma(M;\wedge(T^*M)) \cong \Gamma^+(M;\wedge(T^*M))\oplus\Gamma^-(M;\wedge(T^*M))) \tag{3.50}$$

then the sub-signature operator interchanges $\Gamma^+(M;\wedge(T^*M))$ and $\Gamma^-(M;\wedge(T^*M))$, and $\hat{c}(E,g^E)$ preserves $\Gamma^+(M;\wedge(T^*M))$ and $\Gamma^-(M;\wedge(T^*M))$.

Denotes by $\tilde{D}_E^+$ the restriction of $\tilde{D}_E$ to $\Gamma^+(M,\wedge(T^*M))$. We assume $\dim E = k$ is even, then $(\tilde{D}_E)\hat{c}(E,g^E) = \hat{c}(E,g^E)(\tilde{D}_E)$ and $\hat{c}(E,g^E)$ is a linear map from $\ker\tilde{D}_E^\pm$ to $\ker\tilde{D}_E^\pm$.

The purpose of this section is to compute

$$\mathrm{ind}_{\hat{c}(E,g^E)}[(\tilde{D}_E^+)] = \mathrm{Tr}(\hat{c}(E,g^E)|_{\ker\tilde{D}_E^+}) - \mathrm{Tr}(\hat{c}(E,g^E)|_{\ker\tilde{D}_E^+}). \tag{3.51}$$

By the Mckean-Singer formula, we have

$$\begin{aligned}\mathrm{ind}_{\hat{c}(E,g^E)}(\tilde{D}_E^+) &= \int_M (\sqrt{-1})^{\frac{k}{2}} \mathrm{Tr}[\tilde{\gamma}\hat{c}(E,g^E)k_t(x,\gamma(x))]dx \\ &= \int_M (\sqrt{-1})^{\frac{k}{2}} \mathrm{Tr}[\tilde{\gamma}\hat{c}(E,g^E)K_{(F+\partial_t)^{-1}}(x,\gamma(x),t)]dx.\end{aligned} \tag{3.52}$$

Let

$$R^E - L_1 = \begin{pmatrix} \begin{pmatrix} 0 & -\theta_1 \\ \theta_1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & -\theta_{-\frac{k}{2}} \\ \theta_{-\frac{k}{2}} & 0 \end{pmatrix} \end{pmatrix}, R^{E^\perp} - L_2 = \begin{pmatrix} \begin{pmatrix} 0 & -\hat{\theta}_1 \\ \hat{\theta}_1 & 0 \end{pmatrix} & & 0 & \\ & \ddots & & \\ 0 & & \begin{pmatrix} 0 & -\hat{\theta}_{\frac{n-k-1}{2}} \\ \hat{\theta}_{\frac{n-k-1}{2}} & 0 \end{pmatrix} & \\ & & & 0 \end{pmatrix}; \tag{3.53}$$

and

$$\mathrm{Pf}\left(\frac{R^{E^\perp} - L_2}{2}\right) = \prod_{j=1}^{\frac{n-k-1}{2}} \frac{\hat{\theta}_j}{2}. \tag{3.54}$$

Similar to Theorem 3.9, we get the main Theorem in this section.

Theorem 3.17. *(Local odd dimensional equivariant index Theorem for sub-signature operators)*

Let $x_0 \in M^\gamma$, then

$$\begin{aligned}\lim_{t\to 0} \mathrm{Tr}\left[\tilde{\gamma}(x_0)\hat{c}(E,g^E)I_{(F+\partial_t)^{-1}}(x_0,t)\right] = -\left(\frac{1}{\sqrt{-1}}\right)^{\frac{k}{2}-1} 2^{\frac{n}{2}} \Bigg\{\widehat{A}(R^{M^\gamma})\nu_\phi(R^{N^\gamma})i^*_{M^\gamma}\Bigg[\det{}^{\frac{1}{2}}\left(\cosh\left(\frac{R^E}{4\pi} - \frac{L_1}{2}\right)\right) \\ \times \det{}^{\frac{1}{2}}\left(\frac{\sinh\left(\frac{R^{E^\perp}}{4\pi} - \frac{L_2}{2}\right)}{\frac{R^{E^\perp}}{4\pi} - \frac{L_2}{2}}\right)\mathrm{Pf}\left(\frac{R^{E^\perp}}{4\pi} - \frac{L_2}{2}\right)\Bigg]\Bigg\}^{(a,0)} (x_0).\end{aligned} \tag{3.55}$$

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

AUTHORS' CONTRIBUTION

KB and YW contributed in study conceptualization and writing (review and editing) the manuscript. JW and YW contributed in data curation, formal analysis and writing (original draft). YW contributed in funding acquisition and project administration, supervised the project, formal analysis and writing (original draft) the manuscript.

FUNDING

This research was funded by National Natural Science Foundation of China: No. 11771070. NSFC. 11901322

ACKNOWLEDGMENTS

The work of the first author was supported by NSFC. 11901322. The work of the third author was supported by NSFC. 11771070. The authors also thank the referees for their careful reading and helpful comments.

REFERENCES

[1] M.F. Atiyah, R. Bott, V. Patodi, On the heat equation and the index theorem, Invent. Math. 19 (1973), 279–330.
[2] M.F. Atiyah, I.M. Singer, The index of elliptic operators: I, Ann. of Math. 87 (1968), 484–530.
[3] M.F. Atiyah, I.M. Singer, The index of elliptic operators: III, Ann. of Math. 87 (1968), 546–604.

[4] R. Beals, P.C. Greiner, Calculus on Heisenberg manifolds, Ann. Math. Studies, (AM-119), 119 (1988), 194.
[5] R. Beals, P.C. Greiner, N.K. Stanton, The heat equation on a CR manifold, J. Differential Geom. 20 (1984), 343–387.
[6] N. Berline, E. Getzler, M. Vergne, Heat kernels and Dirac operators, Springer-Verlag, Berlin, 1992.
[7] N. Berline, M. Vergne, A computation of the equivariant index of the Dirac operator, Bull. Soc. Math. France 113 (1985), 305–345.
[8] J.M. Bismut, The Atiyah-Singer index theorem for families of Dirac operators: Two heat equation proofs, Invent. math. 83 (1986), 91–151.
[9] J.M. Bismut, W. Zhang, Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle, Geom. Funct. Anal. 4 (1994), 136–212.
[10] X. Dai, W. Zhang, Adiabatic limit, Bismut-Freed connection, and the real analytic torsion form, J. reine angew. Math. 647 (2010), 87–113.
[11] D. Freed, Two index theorems in odd dimensions, Commu. Anal. Geom. 6 (1998), 317–329.
[12] P.B. Gilkey, Invariance theory, the heat equation, and the Atiyah-Singer index theorem, Publish or Perish, 1984.
[13] P. Greiner, An asymptotic expansion for the heat equation, Arch. Rational Mech. Anal. 41 (1971), 163–218.
[14] J.D. Lafferty, Y.L. Yu, W.P. Zhang, A direct geometric proof of Lefschetz fixed point formulas, Trans. AMS. 329 (1992), 571–583.
[15] K. Liu, X. Ma, On family rigidity theorems, I. Duke Math. J. 102 (2000), 451–474.
[16] K. Liu, Y. Wang, Rigidity theorems on odd dimensional manifolds, Pure and Appl. Math. Quarterly 5 (2009), 1139–1159.
[17] X. Ma, W. Zhang, Eta-invariants, torsion forms and flat vector bundles, Math. Ann. 340 (2008), 569–624.
[18] R. Ponge, A new short proof of the local index formula and some of its applications, Comm. Math. Phys. 241 (2003), 215–234.
[19] R. Ponge, H. Wang, Noncommutative geometry and conformal geometry, II. Connes-Chern character and the local equivariant index theorem, J. Noncomm. Geom. 10 (2016), 307–378.
[20] Y. Wang, Chern-Connes character for the invariant Dirac operator in odd dimensions, Sci. China Ser. A 48 (2005), 1124–1134.
[21] Y. Wang, The Greiner's approach of heat kernel asymptotics and the variation formulas for the equivariant Ray-Singer metric, Int. J. Geom Methods Mod. Phy. 12 (2015), 1550066.
[22] Y. Wang, Volterra calculus, local equivariant family index theorem and equivariant eta forms, Asian J. Math. 20 (2016), 759–784.
[23] Y. Yanlin, Local index theorem for signature operators, Acta Math. Sinica. 3 (1987), 363–372.
[24] W. Zhang, Sub-signature operators, η-invariants and a Riemann-Roch theorem for flat vector bundles, Chin. Ann. Math. 25 (2004), 7–36.
[25] W. Zhang, Sub-signature operator and its local index theorem, Chin. Sci. Bull. 41 (1996), 294–295.
[26] J.W. Zhou, A geometric proof of the Lefschetz fixed-point theorem for signature operators (Chinese), Acta Math. Sinica. 35 (1992), 230–239.

Journal of Nonlinear Mathematical Physics
Vol. **28(3)**; September (2021), *pp.* 321–336
DOI: https://doi.org/10.2991/jnmp.k.210519.001; ISSN 1402-9251; eISSN 1776-0852
https://www.atlantis-press.com/journals/jnmp

Research Article

Analytical Properties for the Fifth Order Camassa-Holm (FOCH) Model

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ARTICLE INFO

Article History

Received 21 March 2021
Accepted 06 May 2021

Keywords

The Fifth order Camassa-Holm (FOCH) model
global existence
infinite propagation speed
long time behavior

2010 Mathematics Subject Classifications

37L05
35Q35
35B44

ABSTRACT

This paper devotes to present analysis work on the fifth order Camassa-Holm (FOCH) model which recently proposed by Liu and Qiao. Firstly, we establish the local and global existence of the solution to the FOCH model. Secondly, we study the property of the infinite propagation speed. Finally, we discuss the long time behavior of the support of momentum density with a compactly supported initial data.

1. INTRODUCTION

In this paper, we consider the following fifth order Camassa-Holm (FOCH) model [29]:

$$\begin{cases} m_t + um_x + bu_x m = 0, & t > 0, x \in \mathbb{R}, \\ m = (1 - \alpha^2 \partial_x^2)(1 - \beta^2 \partial_x^2)u, & t > 0, x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $b \in \mathbb{R}$ is a constant , $\alpha, \beta \in \mathbb{R}$ are two parameters, $\alpha \neq \beta, \alpha\beta \neq 0$. Without loss of generality, we only consider the case $\alpha > 0, \beta > 0$. When $\alpha < 0, \beta < 0$, one can get the similar results by using the corresponding absolute values $|\alpha|$ and $|\beta|$ instead of α, β.

In what follows, we present some mathematical results related to the topic of this paper. Liu and Qiao [29] obtained some interesting solutions including explicit single pseudo-peakons, two-peakon, and N-peakon solutions. Detailed dynamical interactions for two-pseudo-peakons and three-pseudo-peakons were also investigated in their paper with numerical simulations. There have been extensive studies on high order Camassa-Holm type equations in the mathematics physics fields [5,16,17,20,22,31,38–41]. For the case $\alpha = \beta = 1$, on the circle, McLachlan and Zhang [31] established the local well-posedness of the solution in H^s with $s > \frac{7}{2}$, it was shown that system (1.1) with $\alpha = \beta = 1$ does't admit finite time blow-up solutions. Tang and Liu [38] proved that the Cauchy problem for this equation is locally well-posed in the critical Besov space $B^{7/2}_{2,1}$ or $B^s_{p,r}(1 \leq p, r \leq +\infty$ and $s > \max\{3 + \frac{1}{p}, \frac{7}{2}\})$. The peakon-like solution and ill-posedness was also studied in [38]. For the case $b = 2$, $m = u - u_{xx} + u_{xxxx}$, by using the Kato's theory, the local well-posedness [39] was studied in the Sobolev space H^s with $s > \frac{9}{2}$. Ding [16,17] investigated the stationary solution, generality mild traveling solutions and conservative solution. Coclite, Holden and Karlsen [5] established the existence of global weak solutions. They also presented some invariant spaces under the action of the equation. In [20], the infinite propagation speed was considered for the case $m = 4u - 5u_{xx} + u_{xxxx}$. They also proved asymptotic behavior of the solution under the condition that the initial data decays exponentially and algebraically.

When $\beta = 0$ (or $\alpha = 0$), it means $m = u - \alpha^2 u_{xx}$. The Camassa-Holm equation, the Degasperis-Procesi equation, and the Holm-Staley b-family equations are the special cases of equation (1.1) with $b = 2$, $b = 3$ and $b \in \mathbb{R}$, respectively. These equations arise at various levels

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of approximation in shallow water theory, and possess a physics background with shallow water propagation, the bi-Hamiltonian structure, Lax pair, and explicit solutions including classical soliton, cuspon, and peakon solutions.

In 1993, Camassa and Holm [3] derived an integrable shallow water equation with peaked solitons, which was called the Camassa-Holm equation. In 1999, Degasperis and Procesi [15] extended the Camassa-Holm equation to a new water wave equation (Degasperis-Procesi equation). Both Camassa-Holm equation and Degasperis-Procesi equation have attracted much attention. They are completely integrable [11,12,14,35]. Infinitely many conservation laws have been shown in [14,27,37]. For the Camassa-Holm equation, in [9,28], They proved the local well-posedness for the initial datum in H^s with $s > 3/2$. There were many works to study the blow-up phenomenon, such as [8–10,24,28,32]. McKean [32] (See also [24] for a simple proof) proved that if and only if some portion of the positive part of $y_0 = u_0 - u_{0xx}$ lies to the left of some portion of its negative part, then the Camassa-Holm equation blow-up in finite time. The hierarchy properties, related finite-dimensional constrained flows, and algebro-geometric solutions of the Camassa-Holm equation were proposed in [34]. In [1], they studied the global conservative solution for the Camassa-Holm equation. Global dissipative solution have been shown in [2]. Constantin and Strauss [13] studied the orbital stability of the peakons. Himonas and his collaborators [21] obtained the persistence properties and unique continuation of solutions of the Camassa-Holm equation. In [25], the authors deduced the limit of the support of momentum density as t goes to $+\infty$. In [4,6,7,23,26,30,33,35,36,42], they have investigated some mathematical properties for the Degasperis-Procesi equation. For the Holm-Staley b-family equation, mathematical studies have also been presented in [18,19,43].

The paper is organized as follows. In Section 2, we establish the local well-posedness and blow up scenario for the FOCH model. Conditions for global existence are found in Section 3. In Section 4, we establish the property of the infinite propagation speed for the FOCH model. In Section 5, we discuss the long time behavior for the support of momentum density of the FOCH model.

2. LOCAL WELL-POSEDNESS AND BLOW UP SCENARIO

Similar to the Camassa-Holm equation [9], we can establish the following local well-posedness theorem for the FOCH model (1.1).

Theorem 2.1. *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{7}{2}$. Then there exist a $T > 0$ depending on $\|u_0\|_{H^s}$, such that the FOCH model (1.1) has a unique solution*

$$u \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R})).$$

Morever, the map $u_0 \in H^s \to u \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$ is continuous.

The proof is similar to that of Theorem 2.1 in [9,39]. To make the paper concise, we would like to omit the detail proof here. The maximum value of T in Theorem 2.1 is called the lifespan of the solution, in general. If $T < \infty$, that is

$$\lim_{t \to T^-} \|u\|_{H^s} = \infty,$$

we say the solution blows up in finite time.

Before going to the blow up scenario, we have the following Lemma.

Lemma 2.2. *As $m = (1 - \alpha^2\partial_x^2)(1 - \beta^2\partial_x^2)u$, then*

$$u = p * m, \quad p = \frac{\alpha^2}{\alpha^2 - \beta^2} p_1 - \frac{\beta^2}{\alpha^2 - \beta^2} p_2,$$

where $p_1 = \frac{1}{2\alpha} e^{-\frac{|x|}{\alpha}}$, $p_2 = \frac{1}{2\beta} e^{-\frac{|x|}{\beta}}$, $\alpha \neq \beta$, $\alpha > 0$, $\beta > 0$.

Proof. Taking fourier transform to $m = (1 - \alpha^2\partial_x^2)(1 - \beta^2\partial_x^2)u$, we have

$$\hat{m} = (1 + \alpha^2\xi^2)(1 + \beta^2\xi^2)\hat{u}.$$

Notice that when $f(x) = e^{-a|x|}$, $a > 0$ then

$$\hat{f}(\xi) = \frac{2a}{\xi^2 + a^2}.$$

It follows that

$$\hat{u}(\xi) = \frac{1}{1 + \alpha^2\xi^2} \cdot \frac{1}{1 + \beta^2\xi^2} \cdot \hat{m}(\xi) = \hat{p_1}(\xi) \cdot \hat{p_2}(\xi) \cdot \hat{m}(\xi),$$

where $p_1 = \frac{1}{2\alpha} e^{-\frac{|x|}{\alpha}}$, $p_2 = \frac{1}{2\beta} e^{-\frac{|x|}{\beta}}$. Then,

$$u(x) = \mathcal{F}^{-1}\left(\hat{p_1}(\xi) \cdot \hat{p_2}(\xi) \cdot \hat{m}(\xi)\right) = p_1 * p_2 * m(x).$$

Let $p = p_1 * p_2$, we have

$$p(x) = \frac{1}{4\alpha\beta}\int_{\mathbb{R}} e^{-\frac{|x-y|}{\alpha}} e^{-\frac{|y|}{\beta}} dy = \frac{1}{2(\alpha^2-\beta^2)}\left[\alpha e^{-\frac{|x|}{\alpha}} - \beta e^{-\frac{|x|}{\beta}}\right] = \frac{\alpha^2}{(\alpha^2-\beta^2)}p_1 - \frac{\beta^2}{(\alpha^2-\beta^2)}p_2.$$

□

By Lemma 2.2, we can rewrite $u(x,t)$ as

$$\begin{aligned} u &= \left[\frac{\alpha^2}{(\alpha^2-\beta^2)}p_1 - \frac{\beta^2}{(\alpha^2-\beta^2)}p_2\right] * m \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\alpha}}\int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi + e^{\frac{x}{\alpha}}\int_{x}^{+\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi\right] \\ &\quad - \frac{\beta}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^{x} e^{\frac{\xi}{\beta}} m(\xi,t)d\xi + e^{\frac{x}{\beta}}\int_{x}^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right]. \end{aligned} \tag{2.1}$$

Differentiating u with respect to x, we have

$$\begin{aligned} u_x = &\frac{1}{2(\alpha^2-\beta^2)}\left[-e^{-\frac{x}{\alpha}}\int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi + e^{\frac{x}{\alpha}}\int_{x}^{+\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi\right] \\ &+ \frac{1}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^{x} e^{\frac{\xi}{\beta}} m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_{x}^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right]. \end{aligned}$$

Then, we present the precise blow-up scenario.

Theorem 2.3. *Assume that $u_0 \in H^4(\mathbb{R})$ and let T be the maximal existence time of the solution $u(x,t)$ to equation (1.1), $\alpha \neq \beta$, $\alpha > 0$, $\beta > 0$ with the initial data $u_0(x)$.*
(1). *If $b > \frac{1}{2}$, then the corresponding solution of the FOCH model (1.1) blows up in finite time if and only if*

$$\lim_{t\to T}\inf_{x\in\mathbb{R}}\{u_x(x,t)\} = -\infty.$$

(2). *If $b < \frac{1}{2}$, then the corresponding solution of the FOCH model (1.1) blows up in finite time if and only if*

$$\lim_{t\to T}\sup_{x\in\mathbb{R}}\{u_x(x,t)\} = +\infty.$$

Proof. By direct calculation, we have

$$\begin{aligned} \|m\|_{L^2}^2 &= \int_{\mathbb{R}}[u - (\alpha^2+\beta^2)u_{xx} + \alpha^2\beta^2 u_{xxxx}]^2 dx \\ &= \int_{\mathbb{R}} u^2 + (\alpha^2+\beta^2)^2 u_{xx}^2 - 2(\alpha^2+\beta^2)uu_{xx} + \alpha^4\beta^4 u_{xxxx}^2 + 2\alpha^2\beta^2 uu_{xxxx} - 2(\alpha^2+\beta^2)\alpha^2\beta^2 u_{xx}u_{xxxx}dx \\ &= \int_{\mathbb{R}} u^2 + (\alpha^2+\beta^2)^2 u_{xx}^2 + 2(\alpha^2+\beta^2)u_x^2 + \alpha^4\beta^4 u_{xxxx}^2 + 2\alpha^2\beta^2 u_{xx}^2 + 2(\alpha^2+\beta^2)\alpha^2\beta^2 u_{xxx}^2 dx. \end{aligned}$$

Hence

$$c\|u\|_{H^4}^2 \le \|m\|_{L^2}^2 \le C\|u\|_{H^4}^2,$$

where c and C are positive constants depending on α and β. If $b > \frac{1}{2}$, direct calculation we have

$$\frac{d}{dt}\int_{\mathbb{R}} m^2 dx = (1-2b)\int_{\mathbb{R}} u_x m^2 dx \le (1-2b)\inf_{x\in\mathbb{R}}\{u_x(x,t)\}\int_{\mathbb{R}} m^2 dx.$$

If

$$\inf_{x\in\mathbb{R}}\{u_x(x,t)\} \ge -M,$$

then

$$\frac{d}{dt}\int_{\mathbb{R}} m^2 dx \le -(1-2b)M\int_{\mathbb{R}} m^2 dx.$$

By using the Gronwall inequality,

$$\|m\|_{L^2}^2 = \int_{\mathbb{R}} m^2 dx \le e^{-(1-2b)M} \int_{\mathbb{R}} m_0^2 dx = e^{-(1-2b)M} \|m_0\|_{L^2}^2.$$

Therefore the H^4-norm of the solution is bounded on $[0, T)$.

On the other hand,

$$\begin{aligned} u &= \frac{\alpha^2}{(\alpha^2-\beta^2)} p_1 * m - \frac{\beta^2}{(\alpha^2-\beta^2)} p_2 * m \\ &= \frac{\alpha^2}{(\alpha^2-\beta^2)} \int_{\mathbb{R}} p_1(x-\xi) m(\xi) d\xi - \frac{\beta^2}{(\alpha^2-\beta^2)} \int_{\mathbb{R}} p_2(x-\xi) m(\xi) d\xi. \end{aligned}$$

By the Sobolev's embedding $\|u_x\|_\infty \le \|u\|_{H^4}$, it tells us if H^4-norm of the solution is bounded, then the L^∞-norm of the first derivative is bounded.

By the same argument, we can get the similar result for $b < \frac{1}{2}$. So, we omit the details and complete the proof of Theorem 2.3. □

3. GLOBAL EXISTENCE

In this section, we study the global existence. Before going to our main results, we give the particle trajectory as

$$\begin{cases} q_t = u(q,t), & 0 < t < T, x \in \mathbb{R}, \\ q(x,0) = x, & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where T is the lifespan of the solution. Taking derivative (3.1) with respect to x, we obtain

$$\frac{dq_t}{dx} = q_{tx} = u_x(q,t) q_x, \qquad t \in (0,T).$$

Therefore

$$\begin{cases} q_x = exp\{\int_0^t u_x(q,s)ds\}, & 0 < t < T, \quad x \in \mathbb{R}, \\ q_x(x,0) = 1, & x \in \mathbb{R}, \end{cases}$$

which is always positive before the blow-up time. Therefore, the function $q(x,t)$ is an increasing diffeomorphism of the line before blow-up. In fact, direct calculation yields

$$\frac{d}{dt}(m(q)q_x^b) = [m_t(q) + u(q,t)m_x(q) + bu_x(q,t)m(q)]q_x^b = 0.$$

Hence, we have the following identity

$$m(q)q_x^b = m_0(x), \qquad 0 < t < T, x \in \mathbb{R}. \tag{3.2}$$

Theorem 3.1. *Assume that $u_0 \in H^4(\mathbb{R})$, $\alpha \neq \beta$, $\alpha > 0$, $\beta > 0$, if $b = \frac{1}{2}$ or $b = 2$, then the corresponding solution of FOCH model (1.1) will exist globally in time.*

Remark 3.1. *If $\alpha = 0$ or $\beta = 0$, system (1.1) reduce to the well-known b-family equation. The global existence for $b = \frac{1}{2}$ and Theorem 3.2 can be reduce to the results for b-family equation [18]. The global existence for $b = 2$ is the new discovery compared to the b-family equation.*

Proof. Let

$$E(t) = \int_{\mathbb{R}} u^2 + (\alpha^2+\beta^2)u_x^2 + \alpha^2\beta^2 u_{xx}^2 dx.$$

Differentiating $E(t)$, we have

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_{\mathbb{R}} 2uu_t + 2(\alpha^2+\beta^2)u_x u_{xt} + 2\alpha^2\beta^2 u_{xx}u_{xxt} dx \\ &= \int_{\mathbb{R}} 2uu_t - 2(\alpha^2+\beta^2)uu_{xxt} + 2\alpha^2\beta^2 uu_{xxxxt} dx \\ &= 2\int_{\mathbb{R}} um_t dx \\ &= (b-2)\int_{\mathbb{R}} u^2 m_x dx. \end{aligned}$$

It yields that $E(t) = E(0)$ when $b = 2$. By the Sobolev's imbedding, we have

$$\|u_x\|_{L^\infty} \leq \|u\|_{H^2}^2 \leq CE(t) = CE(0).$$

The global existence for $b = 2$ is completed by Theorem 2.3. Applying m on (1.1) and integration by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} m^2 dx &= -2\int_{\mathbb{R}} bu_x m^2 + mm_x u dx\\ &= -2\int_{\mathbb{R}} bu_x m^2 - \frac{m^2}{2}u_x dx\\ &= (1-2b)\int_{\mathbb{R}} u_x m^2 dx.\end{aligned}$$

If $b = \frac{1}{2}$, then $\frac{d}{dt}\int_{\mathbb{R}} m^2 dx = 0$. Hence,

$$\|u\|_{H^4}^2 \leq \|m\|_{L^2}^2 = \|m_0\|_{L^2}^2.$$

It follows that the corresponding solution of FOCH model (1.1) exists globally when $b = \frac{1}{2}$. □

Theorem 3.2. *Supposing that $u_0 \in H^4$, $\alpha \neq \beta$, $\alpha > 0$, $\beta > 0$, $m_0 = (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u_0$ does not change sign. Then the corresponding solution to (1.1) exists globally.*

Proof. We can assume that $m_0 \geq 0$. It is sufficient to prove u_x is bounded for all t. In fact,

$$\begin{aligned}u_x = &\frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{+\infty} e^{-\frac{\xi}{\alpha}}m(\xi,t)d\xi - e^{-\frac{x}{\alpha}}\int_{-\infty}^x e^{\frac{\xi}{\alpha}}m(\xi,t)d\xi\right]\\ &+\frac{1}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^x e^{\frac{\xi}{\beta}}m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}}m(\xi,t)d\xi\right].\end{aligned}$$

If $m_0 \geq 0$, $\alpha > \beta > 0$, then

$$\begin{aligned}u_x = &\frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{+\infty} e^{-\frac{\xi}{\alpha}}m(\xi,t)d\xi - e^{-\frac{x}{\alpha}}\int_{-\infty}^x e^{\frac{\xi}{\alpha}}m(\xi,t)d\xi\right]\\ &+\frac{1}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^x e^{\frac{\xi}{\beta}}m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}}m(\xi,t)d\xi\right]\\ \leq &\frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{\infty} e^{-\frac{\xi}{\alpha}}m(\xi,t)d\xi + e^{-\frac{x}{\beta}}\int_{-\infty}^x e^{\frac{\xi}{\beta}}m(\xi,t)d\xi\right]\\ \leq &\frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}e^{-\frac{x}{\alpha}}\int_x^{\infty} m(\xi,t)d\xi + e^{-\frac{x}{\beta}}e^{\frac{x}{\beta}}\int_{-\infty}^x m(\xi,t)d\xi\right]\\ \leq &\frac{1}{2(\alpha^2-\beta^2)}\left[\int_{\mathbb{R}} m(\xi,t)d\xi + \int_{\mathbb{R}} m(\xi,t)d\xi\right]\\ = &\frac{1}{(\alpha^2-\beta^2)}\int_{\mathbb{R}} m_0(\xi,t)d\xi\end{aligned}$$

and

$$\begin{aligned}u_x = &\frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{\infty} e^{-\frac{\xi}{\alpha}}m(\xi,t)d\xi - e^{-\frac{x}{\alpha}}\int_{-\infty}^x e^{\frac{\xi}{\alpha}}m(\xi,t)d\xi\right]\\ &+\frac{1}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^x e^{\frac{\xi}{\beta}}m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}}m(\xi,t)d\xi\right]\\ \geq &\frac{1}{2(\alpha^2-\beta^2)}\left[-e^{-\frac{x}{\alpha}}\int_{-\infty}^x e^{\frac{\xi}{\alpha}}m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}}m(\xi,t)d\xi\right]\\ \geq &-\frac{1}{(\alpha^2-\beta^2)}\int_{\mathbb{R}} m(\xi,t)d\xi\\ = &-\frac{1}{(\alpha^2-\beta^2)}\int_{\mathbb{R}} m_0(\xi,t)d\xi.\end{aligned}$$

That is

$$|u_x| \leq \frac{1}{(\alpha^2-\beta^2)}\int_{\mathbb{R}} m_0(\xi,t)d\xi.$$

If $m_0 \geq 0$, $0 < \alpha < \beta$, then

$$\begin{aligned} u_x &= \frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi - e^{-\frac{x}{\alpha}}\int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi\right] \\ &\quad + \frac{1}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^{x} e^{\frac{\xi}{\beta}} m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right] \\ &\leq \frac{1}{2(\alpha^2-\beta^2)}\left[-e^{-\frac{x}{\alpha}}\int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right] \\ &\leq \frac{1}{2(\alpha^2-\beta^2)}\left[-e^{-\frac{x}{\alpha}}e^{\frac{x}{\alpha}}\int_{-\infty}^{x} m(\xi,t)d\xi - e^{\frac{x}{\beta}}e^{-\frac{x}{\beta}}\int_x^{\infty} m(\xi,t)d\xi\right] \\ &\leq \frac{1}{2(\beta^2-\alpha^2)}\left[\int_{\mathbb{R}} m(\xi,t)d\xi + \int_{\mathbb{R}} m(\xi,t)d\xi\right] \\ &= \frac{1}{(\beta^2-\alpha^2)}\int_{\mathbb{R}} m_0(\xi,t)d\xi \end{aligned}$$

and

$$\begin{aligned} u_x &= \frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi - e^{-\frac{x}{\alpha}}\int_{-\infty}^{x} e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi\right] \\ &\quad + \frac{1}{2(\alpha^2-\beta^2)}\left[e^{-\frac{x}{\beta}}\int_{-\infty}^{x} e^{\frac{\xi}{\beta}} m(\xi,t)d\xi - e^{\frac{x}{\beta}}\int_x^{+\infty} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi\right] \\ &\geq \frac{1}{2(\alpha^2-\beta^2)}\left[e^{\frac{x}{\alpha}}\int_x^{\infty} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi + e^{-\frac{x}{\beta}}\int_{-\infty}^{x} e^{\frac{\xi}{\beta}} m(\xi,t)d\xi\right] \\ &\geq -\frac{1}{(\beta^2-\alpha^2)}\int_{\mathbb{R}} m(\xi,t)d\xi \\ &= -\frac{1}{(\beta^2-\alpha^2)}\int_{\mathbb{R}} m_0(\xi,t)d\xi. \end{aligned}$$

That is

$$|u_x| \leq \frac{1}{(\beta^2-\alpha^2)}\int_{\mathbb{R}} m_0(\xi,t)d\xi.$$

When $m_0 \leq 0$, via the similar approach that is used above, we could also obtain the global existence result. So, we omit the details and complete the proof of Theorem 3.2. □

4. INFINITE PROPAGATION SPEED

The main theorem reads as follows:

Theorem 4.1. *Assume that the initial datum $u_0(x) \in H^4(\mathbb{R})$ is compactly supported in $[a, c]$, then for $t \in (0, T)$, the corresponding solution $u(x, t)$ to the FOCH model (1.1) $\alpha \neq \beta$, $\alpha > 0$, $\beta > 0$ has the following property:*

$$u(x,t) = \begin{cases} \frac{\alpha}{2(\alpha^2-\beta^2)}e^{-\frac{x}{\alpha}}E_1(t) - \frac{\beta}{2(\alpha^2-\beta^2)}e^{-\frac{x}{\beta}}E_2(t), & \text{as } x > q(c,t), \\ \frac{\alpha}{2(\alpha^2-\beta^2)}e^{\frac{x}{\alpha}}F_1(t) - \frac{\beta}{2(\alpha^2-\beta^2)}e^{\frac{x}{\beta}}F_2(t), & \text{as } x < q(a,t), \end{cases}$$

where

$$E_1(t) = \int_{\mathbb{R}} e^{\frac{x}{\alpha}} m(x,t)dx, \qquad F_1(t) = \int_{\mathbb{R}} e^{-\frac{x}{\alpha}} m(x,t)dx,$$

and

$$E_2(t) = \int_{\mathbb{R}} e^{\frac{x}{\beta}} m(x,t)dx, \qquad F_2(t) = \int_{\mathbb{R}} e^{-\frac{x}{\beta}} m(x,t)dx,$$

denote continuous nonvanishing functions.

Furthermore, if $\alpha > 0$, $0 < \beta \leq \sqrt{\frac{3}{2}}\alpha$, $0 \leq b \leq \min\{3 - \frac{2\beta^2}{\alpha^2}, \frac{5}{3}\}$, $E_1(t)$ is strictly increasing function, while $F_1(t)$ is strictly decreasing function. Similarly, if $\beta > 0$, $0 < \alpha \leq \sqrt{\frac{3}{2}}\beta$, $0 \leq b \leq \min\{3 - \frac{2\alpha^2}{\beta^2}, \frac{5}{3}\}$, $E_2(t)$ is strictly increasing function, while $F_2(t)$ is strictly decreasing function.

Remark 4.1. *Theorem 4.1 implies that the strong solution $u(x,t)$ doesn't have compact x-support for any $t > 0$ in its lifespan, although the corresponding $u_0(x)$ is compactly supported.*

Proof. Since $u_0(x)$ has a compact support in the interval $[a,c]$, so does $m_0(x) = (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u_0(x)$. Equation (3.2) tells us that $m(x) = (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u(x)$ is compactly supported in the interval $[q(a,t), q(c,t)]$ in its lifespan. Hence the following functions are well-defined

$$E_1(t) = \int_{\mathbb{R}} e^{\frac{x}{\alpha}} m(x,t)dx, \qquad F_1(t) = \int_{\mathbb{R}} e^{-\frac{x}{\alpha}} m(x,t)dx,$$
$$E_2(t) = \int_{\mathbb{R}} e^{\frac{x}{\beta}} m(x,t)dx, \qquad F_2(t) = \int_{\mathbb{R}} e^{-\frac{x}{\beta}} m(x,t)dx.$$

Using (3.2),

$$m(q(x,t),t) \equiv 0, \qquad x < a \quad or \quad x > c,$$

we know

$$\begin{aligned} u(x,t) &= \left(\frac{\alpha^2}{(\alpha^2-\beta^2)}p_1 - \frac{\beta^2}{(\alpha^2-\beta^2)}p_2\right) * m(x,t) \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}\int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\alpha}} m(\xi)d\xi - \frac{\beta}{2(\alpha^2-\beta^2)}\int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\beta}} m(\xi)d\xi \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}\int_{q(a,t)}^{q(c,t)} e^{-\frac{|x-\xi|}{\alpha}} m(\xi)d\xi - \frac{\beta}{2(\alpha^2-\beta^2)}\int_{q(a,t)}^{q(c,t)} e^{-\frac{|x-\xi|}{\beta}} m(\xi)d\xi. \end{aligned}$$

Then, for $x > q(c,t)$, we have

$$\begin{aligned} u(x,t) &= \frac{\alpha}{2(\alpha^2-\beta^2)}\int_{q(a,t)}^{q(c,t)} e^{-\frac{x-\xi}{\alpha}} m(\xi)d\xi - \frac{\beta}{2(\alpha^2-\beta^2)}\int_{q(a,t)}^{q(c,t)} e^{-\frac{x-\xi}{\beta}} m(\xi)d\xi \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}e^{-\frac{x}{\alpha}}\int_{q(a,t)}^{q(c,t)} e^{\frac{\xi}{\alpha}} m(\xi)d\xi - \frac{\beta}{2(\alpha^2-\beta^2)}e^{-\frac{x}{\beta}}\int_{q(a,t)}^{q(c,t)} e^{\frac{\xi}{\beta}} m(\xi)d\xi \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}e^{-\frac{x}{\alpha}}E_1(t) - \frac{\beta}{2(\alpha^2-\beta^2)}e^{-\frac{x}{\beta}}E_2(t). \end{aligned} \tag{4.1}$$

Similarly, when $x < q(a,t)$, we have

$$\begin{aligned} u(x,t) &= \frac{\alpha}{2(\alpha^2-\beta^2)}\int_{q(a,t)}^{q(c,t)} e^{\frac{x-\xi}{\alpha}} m(\xi)d\xi - \frac{\beta}{2(\alpha^2-\beta^2)}\int_{q(a,t)}^{q(c,t)} e^{\frac{x-\xi}{\beta}} m(\xi)d\xi \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}e^{\frac{x}{\alpha}}\int_{q(a,t)}^{q(c,t)} e^{-\frac{\xi}{\alpha}} m(\xi)d\xi - \frac{\beta}{2(\alpha^2-\beta^2)}e^{\frac{x}{\beta}}\int_{q(a,t)}^{q(c,t)} e^{-\frac{\xi}{\beta}} m(\xi)d\xi \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)}e^{\frac{x}{\alpha}}F_1(t) - \frac{\beta}{2(\alpha^2-\beta^2)}e^{\frac{x}{\beta}}F_2(t). \end{aligned} \tag{4.2}$$

On the other hand,

$$\frac{dE_1(t)}{dt} = \int_{\mathbb{R}} e^{\frac{\xi}{\alpha}} m_t(\xi,t)d\xi.$$

It is easy to get

$$\begin{aligned} m_t &= -m_x - bmu_x \\ &= [(\alpha^2+\beta^2)u_{xxx} - \alpha^2\beta^2 u_{xxxxx} - u_x]u - b[u - (\alpha^2+\beta^2)u_{xx} + \alpha^2\beta^2 u_{xxxx}]u_x \\ &= (\alpha^2+\beta^2)u_{xxx}u - \alpha^2\beta^2 u_{xxxxx}u - (b+1)uu_x + b(\alpha^2+\beta^2)u_{xx}u_x - b\alpha^2\beta^2 u_{xxxx}u_x. \end{aligned} \tag{4.3}$$

Taking (4.3) into $\frac{dE_1(t)}{dt}$, we obtain

$$\begin{aligned}\frac{dE_1(t)}{dt} &= \int_{\mathbb{R}} e^{\frac{x}{\alpha}} m_t dx \\ &= \int_{\mathbb{R}} e^{\frac{x}{\alpha}} \left[(\alpha^2+\beta^2)u_{xxx}u - \alpha^2\beta^2 u_{xxxxx}u - (b+1)uu_x + b(\alpha^2+\beta^2)u_{xx}u_x - b\alpha^2\beta^2 u_{xxxx}u_x\right] dx \\ &= (\alpha^2+\beta^2)\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xxx}udx - \alpha^2\beta^2 \int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xxxxx}udx - (b+1)\int_{\mathbb{R}} e^{\frac{x}{\alpha}} uu_xdx \\ &\quad + b(\alpha^2+\beta^2)\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xx}u_xdx - b\alpha^2\beta^2\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xxxx}u_xdx \\ &= \sum_{i=1}^{5} I_i. \end{aligned} \tag{4.4}$$

I_1-I_5 can be estimated as follows:

$$\begin{aligned} I_1 &= (\alpha^2+\beta^2)\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xxx}udx \\ &= -\frac{\alpha^2+\beta^2}{2\alpha^3}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u^2dx + \frac{3(\alpha^2+\beta^2)}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_x^2dx, \end{aligned} \tag{4.5}$$

$$\begin{aligned} I_2 &= -\alpha^2\beta^2\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xxxxx}udx \\ &= \frac{\beta^2}{2\alpha^3}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u^2dx - \frac{5\beta^2}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_x^2dx + \frac{5\alpha\beta^2}{2}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xx}^2dx, \end{aligned} \tag{4.6}$$

$$\begin{aligned} I_3 &= -(b+1)\int_{\mathbb{R}} e^{\frac{x}{\alpha}} uu_xdx \\ &= \frac{b+1}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u^2dx, \end{aligned} \tag{4.7}$$

$$\begin{aligned} I_4 &= b(\alpha^2+\beta^2)\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xx}u_xdx \\ &= -\frac{b(\alpha^2+\beta^2)}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_x^2dx, \end{aligned} \tag{4.8}$$

$$\begin{aligned} I_5 &= -b\alpha^2\beta^2\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xxxx}u_xdx \\ &= \frac{b\beta^2}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_x^2dx - \frac{3b\alpha\beta^2}{2}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xx}^2dx. \end{aligned} \tag{4.9}$$

Combining (4.5)–(4.9) to (4.4), we have

$$\begin{aligned}\frac{dE_1(t)}{dt} &= \int_{\mathbb{R}} e^{\frac{x}{\alpha}} m_t dx \\ &= \frac{b}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u^2dx - \frac{(b-3)\alpha^2+2\beta^2}{2\alpha}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_x^2dx + \frac{(5-3b)\alpha\beta^2}{2}\int_{\mathbb{R}} e^{\frac{x}{\alpha}} u_{xx}^2dx. \end{aligned} \tag{4.10}$$

For $\alpha > 0, 0 < \beta \le \sqrt{\frac{3}{2}}\alpha, 0 \le b \le \min\{3 - \frac{2\beta^2}{\alpha^2}, \frac{5}{3}\}$, from (4.10), $E_1(t)$ is strictly increasing for nontrivial solution.

Similary,

$$\begin{aligned}\frac{dF_1(t)}{dt} &= \int_{\mathbb{R}} e^{-\frac{x}{\alpha}} m_t dx \\ &= -\frac{b}{2\alpha}\int_{\mathbb{R}} e^{-\frac{x}{\alpha}} u^2dx + \frac{(b-3)\alpha^2+2\beta^2}{2\alpha}\int_{\mathbb{R}} e^{-\frac{x}{\alpha}} u_x^2dx - \frac{(5-3b)\alpha\beta^2}{2}\int_{\mathbb{R}} e^{-\frac{x}{\alpha}} u_{xx}^2dx. \end{aligned} \tag{4.11}$$

For $\alpha > 0, 0 < \beta \le \sqrt{\frac{3}{2}}\alpha, 0 \le b \le \min\{3 - \frac{2\beta^2}{\alpha^2}, \frac{5}{3}\}$, from (4.11), $F_1(t)$ is strictly decreasing for nontrivial solution.

$$\begin{aligned}\frac{dE_2(t)}{dt} &= \int_{\mathbb{R}} e^{\frac{x}{\beta}} m_t dx \\ &= \frac{b}{2\beta}\int_{\mathbb{R}} e^{\frac{x}{\beta}} u^2dx - \frac{(b-3)\beta^2+2\alpha^2}{2\beta}\int_{\mathbb{R}} e^{\frac{x}{\beta}} u_x^2dx + \frac{(5-3b)\beta\alpha^2}{2}\int_{\mathbb{R}} e^{\frac{x}{\beta}} u_{xx}^2dx. \end{aligned} \tag{4.12}$$

For $\beta > 0, 0 < \alpha \le \sqrt{\frac{3}{2}}\beta, 0 \le b \le \min\{3 - \frac{2\alpha^2}{\beta^2}, \frac{5}{3}\}$, from (4.12), $E_2(t)$ is strictly increasing for nontrivial solution.

$$\begin{aligned}\frac{dF_2(t)}{dt} &= \int_{\mathbb{R}} e^{-\frac{x}{\beta}} m_t dx \\ &= -\frac{b}{2\beta}\int_{\mathbb{R}} e^{-\frac{x}{\beta}} u^2 dx + \frac{(b-3)\beta^2 + 2\alpha^2}{2\beta}\int_{\mathbb{R}} e^{-\frac{x}{\beta}} u_x^2 dx - \frac{(5-3b)\beta\alpha^2}{2}\int_{\mathbb{R}} e^{-\frac{x}{\beta}} u_{xx}^2 dx.\end{aligned} \tag{4.13}$$

For $\beta > 0, 0 < \alpha \le \sqrt{\frac{3}{2}}\beta, 0 \le b \le \min\{3 - \frac{2\alpha^2}{\beta^2}, \frac{5}{3}\}$, from (4.13), $F_2(t)$ is strictly decreasing for nontrivial solution.

This complete the proof of Theorem 4.1. □

Remark 4.2. *Let*

$$u'(x,t) = \begin{cases} \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} E_1(t), & \text{as } x > q(c,t), \\ \frac{\alpha}{2(\alpha^2-\beta^2)} e^{\frac{x}{\alpha}} F_1(t), & \text{as } x < q(a,t), \end{cases} \qquad u''(x,t) = \begin{cases} \frac{\beta}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\beta}} E_2(t), & \text{as } x > q(c,t), \\ \frac{\beta}{2(\alpha^2-\beta^2)} e^{\frac{x}{\beta}} F_2(t), & \text{as } x < q(a,t), \end{cases}$$

We rewrite $u = u' - u''$, as consequences of (4.1) and (4.2), we have

$$\begin{aligned} u'(x,t) &= -\alpha u'_x(x,t) = \alpha^2 u'_{xx}(x,t) = \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} E_1(t), && \text{as } x > q(c,t), \\ u'(x,t) &= \alpha u'_x(x,t) = \alpha^2 u'_{xx}(x,t) = \frac{\alpha}{2(\alpha^2-\beta^2)} e^{\frac{x}{\alpha}} F_1(t), && \text{as } x < q(a,t). \end{aligned}$$

and

$$\begin{aligned} u''(x,t) &= -\beta u''_x(x,t) = \beta^2 u''_{xx}(x,t) = \frac{\beta}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\beta}} E_2(t), && \text{as } x > q(c,t), \\ u''(x,t) &= \beta u''_x(x,t) = \beta^2 u''_{xx}(x,t) = \frac{\beta}{2(\alpha^2-\beta^2)} e^{\frac{x}{\beta}} F_2(t), && \text{as } x < q(a,t). \end{aligned}$$

Theorem 4.2. *Suppose the initial value $u_0(x) \in H^4(\mathbb{R})$, $m_0 = (1-\alpha^2\partial_x^2)(1-\beta^2\partial_x^2)u_0$, $\alpha > \beta > 0$, m_0 doesn't change sign on $\mathbb{R}$ and u_0 has compact support in the interval $[a,c]$. Then for $t \in (0,T)$, the corresponding solution $u(x,t)$ of equation (1.1) satisfies*

$$\begin{aligned} \frac{1}{2(\alpha+\beta)} e^{-\frac{x}{\alpha}} |E_1(t)| \le u(x,t) \le \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} |E_1(t)|, && \text{as } x > q(c,t), \\ \frac{1}{2(\alpha+\beta)} e^{\frac{x}{\alpha}} |F_1(t)| \le u(x,t) \le \frac{\alpha}{2(\alpha^2-\beta^2)} e^{\frac{x}{\alpha}} |F_1(t)|, && \text{as } x < q(a,t). \end{aligned}$$

where

$$E_1(t) = \int_{\mathbb{R}} e^{\frac{\xi}{\alpha}} m(\xi,t) d\xi, \qquad F_1(t) = \int_{\mathbb{R}} e^{-\frac{\xi}{\alpha}} m(\xi,t) d\xi,$$

denote continuous nonvanishing functions.

Remark 4.3. *We assume $\alpha > \beta > 0$ to get the above conclusion in Theorem 4.2, because the position of α, β is symmetric, then $\beta > \alpha > 0$, we have results similar to the above conclusions about $E_2(t) = \int_{\mathbb{R}} e^{\frac{\xi}{\beta}} m(\xi,t)d\xi$, $F_2(t) = \int_{\mathbb{R}} e^{-\frac{\xi}{\beta}} m(\xi,t)d\xi$,*

$$\begin{aligned} \frac{1}{2(\alpha+\beta)} e^{-\frac{x}{\beta}} |E_2(t)| \le u(x,t) \le \frac{\beta}{2(\beta^2-\alpha^2)} e^{-\frac{x}{\beta}} |E_2(t)|, && \text{as } x > q(c,t), \\ \frac{1}{2(\alpha+\beta)} e^{\frac{x}{\beta}} |F_2(t)| \le u(x,t) \le \frac{\beta}{2(\beta^2-\alpha^2)} e^{\frac{x}{\beta}} |F_2(t)|, && \text{as } x < q(a,t). \end{aligned}$$

Theorem 4.2 can be seem as a generalization of the result in [20]. Comparing with Theorem 4.1, it show more detailed estimation by adding the additional condition on m_0.

Proof. If u_0 has a compact support set $[a,c]$, then the corresponding m_0 also has a corresponding compact support set $[a,c]$. It is known from (3.2) that m has the same compact support set $[q(a,t), q(c,t)]$. We define

$$\begin{aligned} u_1 &= \left(1 - \frac{\beta}{\alpha}\right) \cdot \frac{\alpha}{2(\alpha^2-\beta^2)} \int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\alpha}} m d\xi = \frac{1}{2(\alpha+\beta)} \int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\alpha}} m d\xi, \\ u_2 &= \frac{\alpha}{2(\alpha^2-\beta^2)} \int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\alpha}} m d\xi. \end{aligned} \tag{4.14}$$

According to $E_1(t) = \int_{\mathbb{R}} e^{\frac{\xi}{\alpha}} m(\xi,t)d\xi, F_1(t) = \int_{\mathbb{R}} e^{-\frac{\xi}{\alpha}} m(\xi,t)d\xi$, then

$$u_1(x,t) = \frac{1}{2(\alpha+\beta)} e^{-\frac{x}{\alpha}} E_1(t), \qquad u_2(x,t) = \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} E_1(t), \quad \text{as } x > q(c,t),$$
$$u_1(x,t) = \frac{1}{2(\alpha+\beta)} e^{\frac{x}{\alpha}} F_1(t), \qquad u_2(x,t) = \frac{\alpha}{2(\alpha^2-\beta^2)} e^{\frac{x}{\alpha}} F_1(t), \quad \text{as } x < q(a,t).$$

According to (4.1) and (4.14), we obtain

$$u_2(x,t) - u(x,t) = \frac{\beta}{2(\alpha^2-\beta^2)} \int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\beta}} m d\xi,$$
$$u(x,t) - u_1(x,t) = \frac{\beta}{2(\alpha^2-\beta^2)} \int_{\mathbb{R}} \left(e^{-\frac{|x-\xi|}{\alpha}} - e^{-\frac{|x-\xi|}{\beta}} \right) m d\xi.$$

Then, we obtain

$$\begin{cases} u_1(x,t) \le u(x,t) \le u_2(x,t), & m_0 \ge 0, \\ u_2(x,t) \le u(x,t) \le u_1(x,t), & m_0 \le 0. \end{cases}$$

If $m_0 \ge 0$,

$$\begin{cases} \frac{1}{2(\alpha+\beta)} e^{-\frac{x}{\alpha}} E_1(t) \le u(x,t) \le \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} E_1(t), & \text{as } x > q(c,t), \\ \frac{1}{2(\alpha+\beta)} e^{\frac{x}{\alpha}} F_1(t) \le u(x,t) \le \frac{\alpha}{2(\alpha^2-\beta^2)} e^{\frac{x}{\alpha}} F_1(t), & \text{as } x < q(a,t). \end{cases} \tag{4.15}$$

If $m_0 \le 0$,

$$\begin{cases} \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} E_1(t) \le u(x,t) \le \frac{1}{2(\alpha+\beta)} e^{-\frac{x}{\alpha}} E_1(t), & \text{as } x > q(c,t), \\ \frac{\alpha}{2(\alpha^2-\beta^2)} e^{\frac{x}{\alpha}} F_1(t) \le u(x,t) \le \frac{1}{2(\alpha+\beta)} e^{\frac{x}{\alpha}} F_1(t), & \text{as } x < q(a,t). \end{cases} \tag{4.16}$$

The proof of Theorem 4.2 is finished. □

5. LONG TIME BEHAVIOR FOR THE SUPPORT OF MOMENTUM DENSITY

After the global existence of solution is established, we will discuss the long time behavior for the support of momentum density of the FOCH model. Now, we give the lemma and main theorem as follows:

Lemma 5.1. *Let $\alpha > \beta > 0$, Assume the initial value $u_0 \not\equiv 0$ has a compact supported set $[a,c]$.*

(1). *If $m_0(x) \ge 0 (\not\equiv 0)$, $x \in [a,c]$, then we have*

$$\lim_{t\to+\infty} F_1(t) = 0.$$

(2). *If $m_0(x) \le 0 (\not\equiv 0)$, $x \in [a,c]$, then we have*

$$\lim_{t\to+\infty} E_1(t) = 0.$$

Remark 5.1. *By the same argument, we can get a similar conclusion for $\beta > \alpha > 0$. If $m_0(x) \ge 0 (\not\equiv 0)$, $x \in [a,c]$, then we have*

$$\lim_{t\to+\infty} F_2(t) = 0.$$

If $m_0(x) \le 0 (\not\equiv 0)$, $x \in [a,c]$, then we have

$$\lim_{t\to+\infty} E_2(t) = 0.$$

Proof. **(1)** For $m_0(x) > 0$, from (3.2), we have $E_1(t) > 0, F_1(t) > 0, E_2(t) > 0, F_2(t) > 0$, for all $t \ge 0$. As $F_1(t) > 0$, we claim that

$$\lim_{t\to+\infty} F_1(t) = 0.$$

Otherwise, there is a constant $\epsilon_0 > 0$, for any $T > 0$, there will exist a $t > T$, such that $F_1(t) \ge \epsilon_0$.

For any $d < a$, from (4.15) we have

$$\frac{d}{dt} q(d,t) = u(q(d,t),t) \ge \frac{1}{2(\alpha+\beta)} e^{\frac{q(d,t)}{\alpha}} F_1(t)$$
$$\ge \frac{1}{2(\alpha+\beta)} e^{\frac{q(d,t)}{\alpha}} \epsilon_0.$$

It follows that

$$e^{-\frac{q(d,t)}{\alpha}} \le -\frac{\epsilon_0}{2\alpha(\alpha+\beta)}t + \frac{e^{-\frac{d}{\alpha}}}{\alpha}.$$

Taking $T = \frac{2(\alpha+\beta)}{\epsilon_0}e^{-\frac{d}{\alpha}}$, however, when $t = T+1$,

$$-\frac{\epsilon_0}{2\alpha(\alpha+\beta)}t + \frac{e^{-\frac{d}{\alpha}}}{\alpha} < 0,$$

This is the contradiction. So our claim is right.

(2). For $m_0(x) < 0$, from (3.2), we have $E_1(t) < 0, F_1(t) < 0, E_2(t) < 0, F_2(t) < 0$, for all $t \ge 0$. As $F_1(t) > 0$, we claim that

$$\lim_{t\to+\infty} E_1(t) = 0.$$

Otherwise, there is a constant $\epsilon_0 > 0$, for any $T > 0$, for any $T > 0$, there will exist a $t > T$, such that $E_1(t) \le -\epsilon_0$.

For any $h > c$, from (4.16) we have

$$\begin{aligned}\frac{d}{dt}q(h,t) = u(q(h,t),t) &\le \frac{1}{2(\alpha+\beta)}e^{-\frac{q(h,t)}{\alpha}}E_1(t)\\ &\le -\frac{\epsilon_0}{2(\alpha+\beta)}e^{-\frac{q(h,t)}{\alpha}}.\end{aligned}$$

It follows that

$$e^{\frac{q(h,t)}{\alpha}} \le -\frac{\epsilon_0}{2\alpha(\alpha+\beta)}t + \frac{e^{-\frac{d}{\alpha}}}{\alpha}.$$

Taking $T = \frac{2(\alpha+\beta)}{\epsilon_0}e^{-\frac{d}{\alpha}}$, however, when $t = T+1$,

$$-\frac{\epsilon_0}{2\alpha(\alpha+\beta)}t + \frac{e^{-\frac{d}{\alpha}}}{\alpha} < 0,$$

This is the contradiction. So our claim is right. □

Theorem 5.2. *If $b > 1$, $\alpha > \beta > 0$, and suppose that $m_0(x) \in L_{\frac{1}{b}}$ and $u_0(x)$ has a compact supported set $[a,c]$.*

(1). *If $m_0(x) \ge 0 (\not\equiv 0), x \in [a,c]$, then we have*

$$e^{\frac{q(c,t)}{\alpha(b-1)}} - e^{\frac{q(a,t)}{\alpha(b-1)}} \longrightarrow +\infty, \quad \text{as } t \longrightarrow +\infty. \tag{5.1}$$

(2). *If $m_0(x) \le 0 (\not\equiv 0), x \in [a,c]$, then we have*

$$e^{-\frac{q(a,t)}{\alpha(b-1)}} - e^{-\frac{q(c,t)}{\alpha(b-1)}} \longrightarrow +\infty, \quad \text{as } t \longrightarrow +\infty. \tag{5.2}$$

Remark 5.2. *For the case $\beta > \alpha > 0$, by using the properties of E_2 and F_2 in Remark 5.1, one can replace α with β in (5.1) and (5.2).*

Proof. **(1)** By (3.2) and direct calculation, we have

$$\begin{aligned}\left(\int_a^c (m_0)^{\frac{1}{b}}dx\right)^b &= \left(\int_a^c (m(q,t)q_x^b)^{\frac{1}{b}}dx\right)^b = \left(\int_a^c (m(q,t))^{\frac{1}{b}}q_x dx\right)^b\\ &= \left(\int_{q(a,t)}^{q(c,t)} (m(\xi,t))^{\frac{1}{b}}d\xi\right)^b\\ &\le \left(\int_{q(a,t)}^{q(c,t)} m(\xi,t)e^{-\frac{\xi}{\alpha}}d\xi\right)\left[\int_{q(a,t)}^{q(c,t)} e^{\frac{\xi}{\alpha(b-1)}}d\xi\right]^{b-1}\\ &= F_1(t)\left[\alpha(b-1)\left(e^{\frac{q(c,t)}{\alpha(b-1)}} - e^{\frac{q(a,t)}{\alpha(b-1)}}\right)\right]^{(b-1)}.\end{aligned}$$

It follows

$$\left[\alpha(b-1)\left(e^{\frac{q(c,t)}{\alpha(b-1)}}-e^{\frac{q(a,t)(b-1)}{\alpha}}\right)\right]^{(b-1)} \geq \frac{\left(\int_a^c (m_0)^{\frac{1}{b}}dx\right)^b}{F_1(t)}.$$

Using the limit

$$\lim_{t\to+\infty} F_1 = 0,$$

we can get

$$e^{\frac{q(c,t)}{\alpha(b-1)}} - e^{\frac{q(a,t)}{\alpha(b-1)}} \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty.$$

(2). Direct calculation, we have

$$\begin{aligned}\left(\int_a^c(-m_0)^{\frac{1}{b}}dx\right)^b &= \left(\int_a^c(-m_0)^{\frac{1}{b}}dx\right)^b = \left(\int_a^c(-m(q,t)q_x^b)^{\frac{1}{b}}dx\right)^b\\ &= \left(\int_a^c(-m(q,t))^{\frac{1}{b}}q_xdx\right)^b = \left(\int_{q(a,t)}^{q(c,t)}(-m(\xi,t))^{\frac{1}{b}}d\xi\right)^b\\ &\leq \left(\int_{q(a,t)}^{q(c,t)}\left(-m(\xi,t)e^{\frac{\xi}{\alpha}}\right)d\xi\right)\left[\int_{q(a,t)}^{q(c,t)}e^{-\frac{\xi}{\alpha(b-1)}}d\xi\right]^{b-1}\\ &= -E_1(t)\left[\alpha(b-1)(e^{-\frac{q(a,t)}{\alpha(b-1)}}-e^{-\frac{q(c,t)}{\alpha(b-1)}})\right]^{b-1}.\end{aligned}$$

It follows

$$\left[\alpha(b-1)\left(e^{-\frac{q(a,t)}{\alpha(b-1)}}-e^{-\frac{q(c,t)}{\alpha(b-1)}}\right)\right]^{b-1} \geq \frac{\left(\int_a^c(-m_0)^{\frac{1}{b}}dx\right)^b}{-E_1(t)}.$$

Using the limit

$$\lim_{t\to+\infty} E_1 = 0,$$

we can obtain

$$e^{-\frac{q(a,t)}{\alpha(b-1)}} - e^{-\frac{q(c,t)}{\alpha(b-1)}} \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty.$$

□

Theorem 5.3. *If $b = 1$, suppose that $m_0(x) \in L_1$ and $u_0(x)$ has a compact supported set $[a, c]$.*

(1). *If $m_0(x) \geq 0(\not\equiv 0), x \in [a, c]$, then we have*

$$q(c,t) \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty.$$

(2). *If $m_0(x) \leq 0(\not\equiv 0), x \in [a, c]$, then we have*

$$q(a,t) \longrightarrow -\infty, \quad as\ t \longrightarrow +\infty.$$

Proof. We only present the proof for $\alpha > \beta > 0$. The case $\beta > \alpha > 0$ can be proved by the same argument. **(1)** As $m_0(x) \geq 0$, for any $t \geq 0$, we have $F_1(t) > 0$. According to Lemma 5.1, we know

$$\lim_{t\to+\infty} F_1(t) = 0.$$

Direct calculation, we have

$$\int_a^c m_0dx = \int_a^c m(q,t)q_xdx \leq e^{\frac{q(c,t)}{\alpha}}\int_{q(a,t)}^{q(c,t)} m(\xi,t)e^{-\frac{\xi}{\alpha}}d\xi = e^{\frac{q(c,t)}{\alpha}}F_1(t).$$

It follows

$$e^{\frac{q(c,t)}{\alpha}} \geq \frac{\int_a^c m_0dx}{F_1(t)} \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty,$$

then we can get

$$q(c,t) \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty.$$

(2). As $m_0(x) \le 0$, for any $t \ge 0$, we have $E_1(t) < 0$. According to Lemma 5.1, we know

$$\lim_{t\to+\infty} E_1(t) = 0.$$

Direct calculation, we have

$$\int_a^c (-m_0)dx = \int_a^c (-m(q,t)q_x)dx$$
$$\le e^{-\frac{q(a,t)}{\alpha}} \int_{q(a,t)}^{q(c,t)} (-m(\xi,t))e^{\frac{\xi}{\alpha}} d\xi = -e^{-\frac{q(a,t)}{\alpha}} E_1(t).$$

It follows

$$e^{-\frac{q(a,t)}{\alpha}} \ge \frac{\int_a^c(-m_0)dx}{-E_1(t)} \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty,$$

then we can get

$$q(a,t) \longrightarrow -\infty, \quad as\ t \longrightarrow +\infty.$$

□

Theorem 5.4. *If $0 < b < 1$, $\alpha > \beta > 0$, $m_0(x) \in L_{\frac{1}{b}}$ or $b = 0$, $m_0 \in L_\infty$. Suppose that $u_0(x)$ has a compact supported set $[a,c]$.*

***(1).** If $m_0(x) \ge 0 (\not\equiv 0)$ for $x \in [a,c]$, then we have*

$$e^{\frac{q(c,t)}{\alpha}} - e^{\frac{q(a,t)}{\alpha}} \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty. \tag{5.3}$$

***(2).** If $m_0(x) \le 0 (\not\equiv 0)$ for $x \in [a,c]$, then we have*

$$e^{-\frac{q(a,t)}{\alpha}} - e^{-\frac{q(c,t)}{\alpha}} \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty. \tag{5.4}$$

Remark 5.3. *For the case $\beta > \alpha > 0$, by using the properties of E_2 and F_2 in Remark 5.1, one can replace α with β in (5.3) and (5.4).*

Proof. **(1).** For $m_0(x) \ge 0$, we have $F_1(t) > 0$ for all $t \ge 0$. From Lemma 5.1, we know

$$\lim_{t\to+\infty} F_1(t) = 0.$$

According to the conservation law

$$\int_{\mathbb{R}} m dx = \int_{\mathbb{R}} m_0 dx, \qquad \int_{\mathbb{R}} m^{\frac{1}{b}} dx = \int_{\mathbb{R}} m_0^{\frac{1}{b}} dx.$$

$$If\ \ 0 < b < 1\ \ and\ \ \begin{cases} \gamma + \frac{\eta}{b} = 1, \\ 2\gamma + \eta = 1, \\ 0 < \gamma, \eta < 1. \end{cases} \Longrightarrow \begin{cases} 0 < \eta = \frac{2}{2-b} - 1 < 1, \\ 0 < \gamma = 1 + \frac{1}{b-2} < 1. \end{cases}$$

By direct calculation, we obtain

$$\int_{\mathbb{R}} m_0 dx = \int_{\mathbb{R}} m dx = \int_{\mathbb{R}} m(q,t)q_x dx$$
$$= \left[\int_a^c \left(me^{-\frac{q}{\alpha}}q_x\right)^\gamma \left(m^{\frac{1}{b}}q_x\right)^\eta \left(e^{\frac{q}{\alpha}}q_x\right)^\gamma dx\right]$$
$$\le \left(\int_a^c me^{-\frac{q}{\alpha}}q_x dx\right)^\gamma \left(\int_a^c m^{\frac{1}{b}}q_x dx\right)^\eta \left(\int_a^c e^{\frac{q}{\alpha}}q_x dx\right)^\gamma$$
$$= \left(\int_{q(a,t)}^{q(c,t)} me^{-\frac{\xi}{\alpha}} d\xi\right)^\gamma \left(\int_{\mathbb{R}} m^{\frac{1}{b}} d\xi\right)^\eta \left(\int_{q(a,t)}^{q(c,t)} e^{\frac{\xi}{\alpha}} d\xi\right)^\gamma$$
$$= (F_1(t))^\gamma \left(\int_{\mathbb{R}} m^{\frac{1}{b}} d\xi\right)^\eta \left(\int_{q(a,t)}^{q(c,t)} e^{\frac{\xi}{\alpha}} d\xi\right)^\gamma$$
$$= (F_1(t))^\gamma \left(\int_{\mathbb{R}} m_0^{\frac{1}{b}} d\xi\right)^\eta \left(\alpha e^{\frac{q(c,t)}{\alpha}} - \alpha e^{\frac{q(a,t)}{\alpha}}\right)^\gamma.$$

It follows

$$\left(\alpha e^{\frac{q(c,t)}{\alpha}}-\alpha e^{\frac{q(a,t)}{\alpha}}\right)^{\gamma}\geq\frac{\int_{\mathbb{R}}m_0dx}{(F_1(t))^{\gamma}\left(\int_{\mathbb{R}}m_0^{\frac{1}{b}}d\xi\right)^{\eta}}\longrightarrow+\infty,$$

then we can obtain

$$e^{\frac{q(c,t)}{\alpha}}-e^{\frac{q(a,t)}{\alpha}}\longrightarrow+\infty,\quad as\ t\longrightarrow+\infty.$$

If $b=0$, we can obtain

$$\begin{aligned}\int_{\mathbb{R}}m_0dx&=\lim_{b\to0}\int_{\mathbb{R}}mdx=\int_{\mathbb{R}}m(q,t)q_xdx\\&=\lim_{b\to0}\left[\int_a^c\left(me^{-\frac{q}{\alpha}}q_x\right)^{\gamma}\left(m^{\frac{1}{b}}q_x\right)^{\eta}\left(e^{\frac{q}{\alpha}}q_x\right)^{\gamma}dx\right]\\&\leq\lim_{b\to0}\left(\int_a^c me^{-\frac{q}{\alpha}}q_xdx\right)^{\gamma}\left(\int_a^c m^{\frac{1}{b}}q_xdx\right)^{\eta}\left(\int_a^c e^{\frac{q}{\alpha}}q_xdx\right)^{\gamma}\\&=\lim_{b\to0}\left(\int_{q(a,t)}^{q(c,t)}me^{-\frac{\xi}{\alpha}}d\xi\right)^{\gamma}\left(\int_{\mathbb{R}}m^{\frac{1}{b}}d\xi\right)^{\eta}\left(\int_{q(a,t)}^{q(c,t)}e^{\frac{\xi}{\alpha}}d\xi\right)^{\gamma}\\&=\lim_{b\to0}(F_1(t))^{\gamma}\left(\int_{\mathbb{R}}m^{\frac{1}{b}}d\xi\right)^{\eta}\left(\int_{q(a,t)}^{q(c,t)}e^{\frac{\xi}{\alpha}}d\xi\right)^{\gamma}\\&=\lim_{b\to0}(F_1(t))^{\gamma}\left(\int_{\mathbb{R}}m_0^{\frac{1}{b}}d\xi\right)^{\eta}\left(\alpha e^{\frac{q(c,t)}{\alpha}}-\alpha e^{\frac{q(a,t)}{\alpha}}\right)^{\gamma}.\end{aligned}$$

It follows

$$\left(\alpha e^{\frac{q(c,t)}{\alpha}}-\alpha e^{\frac{q(a,t)}{\alpha}}\right)^{\gamma}\geq\frac{\int_{\mathbb{R}}m_0dx}{(F_1(t))^{\gamma}\left(\lim_{b\to0}\int_{\mathbb{R}}m_0^{\frac{1}{b}}d\xi\right)^{\eta}}\longrightarrow+\infty,$$

then we can obtain

$$e^{\frac{q(c,t)}{\alpha}}-e^{\frac{q(a,t)}{\alpha}}\longrightarrow+\infty,\quad as\ t\longrightarrow+\infty.$$

(2). For $m_0(x)\leq0$, we have $E_1(t)<0$ for all $t\geq0$. From Lemma 5.1, we know

$$\lim_{t\to+\infty}E_1(t)=0.$$

Similarly, according to the conservation law

$$\int_{\mathbb{R}}mdx=\int_{\mathbb{R}}m_0dx,\qquad\int_{\mathbb{R}}m^{\frac{1}{b}}dx=\int_{\mathbb{R}}m_0^{\frac{1}{b}}dx.$$

$$\text{If } 0<b<1 \text{ and }\begin{cases}\gamma+\frac{\eta}{b}=1,\\2\gamma+\eta=1,\\0<\gamma,\eta<1.\end{cases}\Longrightarrow\begin{cases}0<\eta=\frac{2}{2-b}-1<1,\\0<\gamma=1+\frac{1}{b-2}<1.\end{cases}$$

By direct calculation, we obtain

$$\begin{aligned}-\int_{\mathbb{R}}m_0dx&=-\int_{\mathbb{R}}mdx=-\int_{\mathbb{R}}m(q,t)q_xdx\\&=\left[\int_a^c\left(-me^{\frac{q}{\alpha}}q_x\right)^{\gamma}\left((-m)^{\frac{1}{b}}q_x\right)^{\eta}\left(e^{-\frac{q}{\alpha}}q_x\right)^{\gamma}dx\right]\\&\leq\left(\int_a^c -me^{\frac{q}{\alpha}}q_xdx\right)^{\gamma}\left(\int_a^c(-m)^{\frac{1}{b}}q_xdx\right)^{\eta}\left(\int_a^c e^{-\frac{q}{\alpha}}q_xdx\right)^{\gamma}\\&=\left(\int_{q(a,t)}^{q(c,t)}-me^{\frac{\xi}{\alpha}}d\xi\right)^{\gamma}\left(\int_{\mathbb{R}}(-m)^{\frac{1}{b}}d\xi\right)^{\eta}\left(\int_{q(a,t)}^{q(c,t)}e^{-\frac{\xi}{\alpha}}d\xi\right)^{\gamma}\end{aligned}$$

$$= (-E_1(t))^{\gamma}\left(\int_{\mathbb{R}} m^{\frac{1}{b}}d\xi\right)^{\eta}\left(\int_{q(a,t)}^{q(c,t)} e^{-\frac{\xi}{\alpha}}d\xi\right)^{\gamma}$$
$$= (-E_1(t))^{\gamma}\left(\int_{\mathbb{R}} m_0^{\frac{1}{b}}d\xi\right)^{\eta}\left(\alpha e^{-\frac{q(a,t)}{\alpha}} - \alpha e^{-\frac{q(c,t)}{\alpha}}\right)^{\gamma}.$$

It follows

$$\left(\alpha e^{-\frac{q(a,t)}{\alpha}} - \alpha e^{-\frac{q(c,t)}{\alpha}}\right)^{\gamma} \geq \frac{-\int_{\mathbb{R}} m_0 dx}{(-E_1(t))^{\gamma}\left(\int_{\mathbb{R}}(-m_0)^{\frac{1}{b}}d\xi\right)^{\eta}} \longrightarrow +\infty,$$

then we can obtain

$$e^{-\frac{q(a,t)}{\alpha}} - e^{-\frac{q(c,t)}{\alpha}} \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty.$$

If $b = 0$, we can obtain

$$-\int_{\mathbb{R}} m_0 dx = -\lim_{b\to 0}\int_{\mathbb{R}} m dx = -\int_{\mathbb{R}} m(q,t)q_x dx$$
$$= \lim_{b\to 0}\left[\int_a^c \left(-me^{\frac{q}{\alpha}}q_x\right)^{\gamma}\left((-m)^{\frac{1}{b}}q_x\right)^{\eta}\left(e^{-\frac{q}{\alpha}}q_x\right)^{\gamma}dx\right]$$
$$\leq \lim_{b\to 0}\left(\int_a^c -me^{\frac{q}{\alpha}}q_x dx\right)^{\gamma}\left(\int_a^c(-m)^{\frac{1}{b}}q_x dx\right)^{\eta}\left(\int_a^c e^{-\frac{q}{\alpha}}q_x dx\right)^{\gamma}$$
$$= \lim_{b\to 0}\left(\int_{q(a,t)}^{q(c,t)} -me^{\frac{\xi}{\alpha}}d\xi\right)^{\gamma}\left(\int_{\mathbb{R}}(-m)^{\frac{1}{b}}d\xi\right)^{\eta}\left(\int_{q(a,t)}^{q(c,t)} e^{-\frac{\xi}{\alpha}}d\xi\right)^{\gamma}$$
$$= \lim_{b\to 0}(-E_1(t))^{\gamma}\left(\int_{\mathbb{R}} m^{\frac{1}{b}}d\xi\right)^{\eta}\left(\int_{q(a,t)}^{q(c,t)} e^{-\frac{\xi}{\alpha}}d\xi\right)^{\gamma}$$
$$= \lim_{b\to 0}(-E_1(t))^{\gamma}\left(\int_{\mathbb{R}} m_0^{\frac{1}{b}}d\xi\right)^{\eta}\left(\alpha e^{-\frac{q(a,t)}{\alpha}} - \alpha e^{-\frac{q(c,t)}{\alpha}}\right)^{\gamma}.$$

It follows

$$\left(\alpha e^{-\frac{q(a,t)}{\alpha}} - \alpha e^{-\frac{q(c,t)}{\alpha}}\right)^{\gamma} \geq \frac{-\int_{\mathbb{R}} m_0 dx}{(-E_1(t))^{\gamma}\left(\lim_{b\to 0}\int_{\mathbb{R}}(-m_0)^{\frac{1}{b}}d\xi\right)^{\eta}} \longrightarrow +\infty,$$

then we can obtain

$$e^{-\frac{q(a,t)}{\alpha}} - e^{-\frac{q(c,t)}{\alpha}} \longrightarrow +\infty, \quad as\ t \longrightarrow +\infty. \qquad \square$$

6. CONCLUSION

We have considered the FOCH model $\alpha \neq \beta, \alpha > 0, \beta > 0$. When α, β is negative, one can get the same results by taking absolute value $|\alpha|$ and $|\beta|$. This model is highly related to the classical Camassa-Holm equation, the Degasperis-Procesi equation and the Holm-Staley b-family equation. We have studied some mathematical property, such as global existence, infinite propagation speed and long time behavior of the support of momentum density. Another highly related equation is (1.1) with $\alpha = \beta$. Due to (1.1) with $\alpha = \beta$ doesn't have the structure in Lemma 2.2 and (2.1), some results in this manuscript may can't been realized for $\alpha = \beta$.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

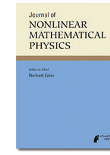
ACKNOWLEDGEMENTS

This work has partially been supported by the National Natural Science Foundation of China (No. 12071439 and No. 11971475) and the 2019 - 2020 Hunan overseas distinguished professorship project (No. 2019014). The third author Jiang was partially supported by NSFC (grant No. 11101376, 11671309, 11671364) and ZJNSF (grant No. LY19A010016). The fourth author Qiao thanks the UT President's Endowed Professorship (Project # 450000123) for its partial support.

REFERENCES

[1] A. Bressan, A. Constantin, Global conservative solutions of the Camassa–Holm equation, Arch. Ration. Mech. Anal. 183 (2007), 215–239.
[2] A. Bressan, A. Constantin, Global dissipative solutions of the Camassa-Holm equation, Anal. Appl. 5 (2007), 1–27.
[3] R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993), 1661–1664.
[4] O. Christov, S. Hakkaev, On the Cauchy problem for the periodic b-family of equations and of the non-uniform continuity of Degasperis–Procesi equation, J. Math. Anal. Appl. 360 (2009), 47–56.
[5] G.M. Coclite, H. Holden, K.H. Karlsen, Well-posedness of higher-order Camassa–Holm equations, J. Diff. Eqs. 246 (2009), 929–963.
[6] G.M. Coclite, K.H. Karlsen, Periodic solutions of the Degasperis–Procesi equation: well-posedness and asymptotics, J. Funct. Anal. 268 (2015), 1053–1077.
[7] G.M. Coclite, K.H. Karlsen, Y.S. Kwon, Initial–boundary value problems for conservation laws with source terms and the Degasperis–Procesi equation, J. Funct. Anal. 257 (2009), 3823–3857.
[8] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: a geometric approach, Ann. Inst. Fourier (Grenoble) 50 (2000), 321–362.
[9] A. Constantin, J. Escher, Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation, Comm. Pure Appl. Math. 51 (1998), 475–504.
[10] A. Constantin, J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, Acta Math. 181 (1998), 229–243.
[11] A. Constantin, V.S. Gerdjikov, R.I. Ivanov, Inverse scattering transform for the Camassa–Holm equation, Inver. Prob. 22 (2006), 2197–2207.
[12] A. Constantin, H.P. McKean, A shallow water equation on the circle, Comm. Pure Appl. Math. 52 (1999), 949–982.
[13] A. Constantin, W.A. Strauss, Stability of peakons, Comm. Pure Appl. Math. 53 (2000), 603–610.
[14] A. Degasperis, D.D. Holm, A.N.I. Khon, A new integrable equation with peakon solutions, Teoret. Mat. Fiz. 133 (2002), 170–183.
[15] A. Degasperis, M. Procesi, Asymptotic integrability, in: A. Degasperis, G. Gaeta (Eds.), Symmetry and Perturbation Theory, World Scientific, Singapore, 1999, pp. 23–37.
[16] D. Ding, Traveling solutions and evolution properties of the higher order Camassa–Holm equation, Nonlinear Anal: Theory Meth. Appl. 152 (2017), 1–11.
[17] D. Ding, P. Lv, Conservative solutions for higher-order Camassa-Holm equations, J. Math. Phys. 51 (2010), 072701.
[18] J. Escher, Z. Yin, Well-posedness, blow-up phenomena, and global solutions for the b-equation, J. Reine Angew. Math. 624 (2008), 51–80.
[19] G. Gui, Y. Liu, L. Tian, Global existence and blow-up phenomena for the peakon b-family of equations, Indiana Univ. Math. J. 57 (2008), 1209–1234.
[20] L. Han, W. Cui, Infinite propagation speed and asymptotic behavior for a generalized fifth-order Camassa–Holm equation, Appl. Anal. 98 (2019), 536–552.
[21] A.A. Himonas, G. Misiołek, G. Ponce, Y. Zhou, Persistence properties and unique continuation of solutions of the Camassa-Holm equation, Comm. Math. Phys. 271 (2007), 511–522.
[22] D.D. Holm, A.N.W. Hone, Nonintegrability of a fifth-order equation with integrable two-body dynamics, Theoret. Math. Phys. 137 (2003), 1459–1471.
[23] Y. Hou, P. Zhao, E. Fan, Z. Qiao, Algebro-geometric solutions for the Degasperis–Procesi hierarchy, SIAM J. Math. Anal. 45 (2013), 1216–1266.
[24] Z. Jiang, L. Ni, Y. Zhou, Wave breaking of the Camassa–Holm equation, J. Nonlinear Sci. 22 (2012), 235–245.
[25] Z. Jiang, Y. Zhou, M. Zhu, Large time behavior for the support of momentum density of the Camassa-Holm equation, J. Math. Phys. 54 (2013), 081503.
[26] J. Lenells, Traveling wave solutions of the Degasperis–Procesi equation, J. Math. Anal. Appl. 306 (2005), 72–82.
[27] J. Lenells, Conservation laws of the Camassa–Holm equation, J. Phys. A: Math. Gen. 38 (2005), 869–880.
[28] Y.A. Li, P.J. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, J. Diff. Eqs. 162 (2000), 27–63.
[29] Q. Liu, Z. Qiao, Fifth order Camassa-Holm model with pseudo-peakons and multi-peakons, Int. J. Non-Linear Mech. 105 (2018), 179–185.
[30] Y. Liu, Z. Yin, Global existence and blow-up phenomena for the Degasperis-Procesi equation, Comm. Math. Phys. 267 (2006), 801–820.
[31] R. McLachlan, X. Zhang, Well-posedness of modified Camassa–Holm equations, J. Diff. Eqs. 246 (2009), 3241–3259.
[32] H.P. McKean, Breakdown of a shallow water equation, Asian J. Math. 2 (1998), 867–874.
[33] O.G. Mustafa, A note on the Degasperis-Procesi equation, J. Nonlinear Math. Phys. 12 (2005), 10–14.
[34] Z. Qiao, The Camassa-Holm hierarchy, N-dimensional integrable systems, and algebro-geometric solution on a symplectic submanifold, Comm. Math. Phys. 239 (2003), 309–341.
[35] Z. Qiao, Integrable hierarchy, 3×3 constrained systems, and parametric solutions, Acta Appl. Math. 83 (2004), 199–220.
[36] Z. Qiao, S. Li, A new integrable hierarchy, parametric solutions, and traveling wave solutions, Math. Phys. Anal. Geom. 7 (2004), 289–308.
[37] E.G. Reyes, Geometric integrability of the Camassa–Holm equation, Lett. Math. Phys. 59 (2002), 117–131.
[38] H. Tang, Z. Liu, Well-posedness of the modified Camassa–Holm equation in Besov spaces, Z. Angew. Math. Phys. 66 (2015), 1559–1580.
[39] L. Tian, P. Zhang, L. Xia, Global existence for the higher-order Camassa–Holm shallow water equation, Nonlinear Anal: Theory Meth. Appl. 74 (2011), 2468–2474.
[40] F. Wang, F. Li, Z. Qiao, Well-posedness and peakons for a higher-order μ-Camassa–Holm equation, Nonlinear Anal. 175 (2018), 210–236.
[41] F. Wang, F. Li, Z. Qiao, On the Cauchy problem for a higher-order μ-Camassa–Holm equation, Discrete Contin. Dyn. Syst. 38 (2018), 4163–4187.
[42] Y. Zhou, Blow-up phenomenon for the integrable Degasperis–Procesi equation, Phys. Lett. A 328 (2004), 157–162.
[43] Y. Zhou, On solutions to the Holm Staley b-family of equations, Nonlinearity 23 (2010), 369–381.

Journal of Nonlinear Mathematical Physics
Vol. **28(3)**; September (2021), *pp*. 337–343
DOI: https://doi.org/10.2991/jnmp.k.210614.001; ISSN 1402-9251; eISSN 1776-0852
https://www.atlantis-press.com/journals/jnmp



Research Article

On the Generalized KdV Hierarchy and Boussinesq Hierarchy with Lax Triple

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ARTICLE INFO

Article History

Received 11 March 2021
Accepted 31 May 2021

Keywords

Lax equation
KdV hierarchy
Boussinesq hierarchy
Lax triple

2000 Mathematics Subject Classifications

37K10
35Q53

ABSTRACT

Based on the Nambu 3-bracket and the operators of the KP hierarchy, we propose the generalized Lax equation of the Lax triple. Under the operator constraints, we construct the generalized KdV hierarchy and Boussinesq hierarchy. Moreover, we present the exact solutions of some nonlinear evolution equations.

1. INTRODUCTION

Nambu mechanics [8] is a generalization of classical Hamiltonian mechanics based on Liouville theorem. Poisson brackets in Hamilton mechanics are replaced by Nambu brackets. Based on Nambu brackets, Nambu 3-algebra [10] is introduced. It is a natural generalization of Lie Algebra with high structure. 3-algebra has been widely applied in string theory and M-branches [1,9]. In recent years, the relationship between infinite dimensional 3-algebra and integrable system has attracted wide attention in the framework of Nambu mechanics [2,3,14].

The Kadomtsev-Petviashvili (KP) hierarchy [5–7,13] is an important classical integrable system. There are different approaches to the description of the KP hierarchy. One of them is described in terms of a Lax pair (B_n, L). By means of the operator Nambu 3-bracket, the generalized Lax equation of the KP hierarchy with the Lax triple (B_m, B_n, L) was studied in [12], where the KP equation and other integrable (nonintegrable) equations were derived, and the soliton wave solutions of the nonlinear evolution equations were provided. The BKP and CKP hierarchies are two important reductions of the KP hierarchy. When the operator L satisfies the constraints $L^* = -\partial L \partial^{-1}$ and $L^* = -L$, the KP hierarchy becomes the BKP and CKP hierarchies, respectively. The dKP hierarchy is the quasi classical limit of the KP hierarchy. Based on the Lax triple (B_m, B_n, L), the generalized BKP, CKP and dKP hierarchies were investigated [4,11]. When the operator L satisfies the constraints $(L^2)_- = 0$ and $(L^3)_- = 0$, the KP hierarchy becomes the KdV and Boussinesq hierarchies, respectively. Both KdV equation and Boussinesq equation are derived from the study of shallow water waves. They both contain N-soliton solutions. Boussinesq equation can be considered as a generalization of KdV equation, which allows solitons to propagate in two directions. The aim of this paper is to derive the nonlinear evolution equations from the generalized Lax equation in term of the Lax triple (B_m, B_n, L) of the KdV and Boussinesq hierarchies.

This paper is arranged as follows. In Section 2, the generalized Lax equation of KP hierarchy and operator constraints is introduced. In Sections 3 and 4, we give the generalized KdV and Boussinesq hierarchies, respectively. Finally, a short conclusion and further discussion are presented.

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Data availability statement: The data that support the findings of this study are available from the corresponding author [XW], upon reasonable request.

2. GENERALIZED LAX EQUATION

The KP hierarchy can be derived from the well-known Lax equation,

$$\frac{\partial L}{\partial t_n} = [B_n, L] = B_n L - L B_n, \quad n = 1, 2, \cdots . \tag{2.1}$$

Here $B_n = (L^n)_+, \ n \geq 1$. L is a pseudo-differential operator,

$$L = \partial + \sum_{i=0}^{+\infty} v_i(t)\partial^{-i-1}, \tag{2.2}$$

where $t = (t_1, t_2, \cdots)$ are the time variables and $\partial = \partial/\partial_x$, $x = t_1$, the negative powers of ∂ are to be understood as the formal integration symbols.

As the operator L satisfies the constraints $(L^2)_- = 0$ and $(L^3)_- = 0$, respectively, we can derive the KdV hierarchy and Boussinesq hierarchy from the Lax equation (2.1). Here $(L^k)_-$, $k = 2, 3$, denotes the integral part of L^k,

$$(L^k)_- = L^k - (L^k)_+ = L^k - B_k.$$

The constraints $(L^k)_- = 0$ means $[B_{kn}, B_k] = 0, n = 1, 2, \ldots$, thus we can derive

$$\frac{\partial L^k}{\partial t_{kn}} = 0, k = 2, 3, n = 1, 2, \ldots . \tag{2.3}$$

Based on the operator Nambu 3-bracket, the generalized Lax equation with respect to the Lax triple (L, B_n, B_m)[12] is defined by

$$\frac{\partial L}{\partial t_{mn}} = [B_m, B_n, L]_-, \quad (m, n = 0, 1, 2 \cdots), \tag{2.4}$$

where $B_0 = 1$. The operator Nambu 3-bracket $[,,]_-$ denotes the formal integration operator part of the derived pseudo-differential operator.

Taking $B_m = B_0$ in (2.4), it is easy to verify that (2.4) leads to the Lax equation (2.1). Thus it is natural to derive the KP hierarchy from (2.4). As the operator L satisfy the constraints $(L^2)_- = 0$ and $(L^3)_- = 0$, respectively, we can also derive the KdV hierarchy and Boussinesq hierarchy.

In the following, we will list the usual KdV hierarchy and Boussinesq hierarchy. And we also will derive the generalized KdV hierarchy and Boussinesq hierarchy from the generalized Lax equation (2.4).

3. GENERALIZED KdV HIERARCHY

Equating the coefficients of the operator $\partial^{-i}(i = 1, 2, \cdots)$ in the constraints $(L^2)_- = 0$, we can derive

$$\begin{aligned}
v_1 &= -\frac{1}{2}v_{0,x}, \\
v_2 &= -\frac{1}{2}v_0^2 + \frac{1}{4}v_{0,xx}, \\
v_3 &= \frac{3}{2}v_0 v_{0,x} - \frac{1}{8}v_{0,xxx}, \\
v_4 &= -\frac{7}{4}v_0 v_{0,xx} + \frac{1}{2}v_0^3 - \frac{11}{8}v_{0,x}^2 + \frac{1}{16}v_{0,xxxx}, \\
v_5 &= \frac{15}{8}v_0 v_{0,xxx} + \frac{15}{4}v_{0,x}v_{0,xx} - \frac{15}{4}v_{0,x}v_0^2 - \frac{1}{32}v_{0,xxxxx}, \\
&\vdots
\end{aligned} \tag{3.1}$$

Then we can derive B_n of the KdV hierarchy are

$$\begin{aligned}
B_1 &= \partial, \\
B_2 &= \partial^2 + 2v_0, \\
B_3 &= \partial^3 + 3v_0\partial + \frac{3}{2}v_{0,x}, \\
B_4 &= \partial^4 + 4v_0\partial^2 + 4v_{0,x}\partial + 4v_0^2 + 2v_{0,xx}, \\
B_5 &= \partial^5 + 5v_0\partial^3 + \frac{15}{2}v_{0,x}\partial^2 + \left(\frac{15}{2}v_0^2 + \frac{25}{4}v_{0,xx}\right)\partial + \frac{15}{2}v_0 v_{0,x} + \frac{15}{8}v_{0,xxx},
\end{aligned}$$

$$B_6 = \partial^6 + 6v_0\partial^4 + 12v_{0,x}\partial^3 + (12v_0^2 + 14v_{0,xx})\partial^2 + (24v_0v_{0,x} + 8v_{0,xxx})\partial + 2v_{0,xxxx} + 12v_0v_{0,xx} + 8v_0^3 + 8v_{0,x}^2,$$

$$B_7 = \partial^7 + 7v_0\partial^5 + \frac{35}{2}v_{0,x}\partial^4 + \left(\frac{35}{2}v_0^2 + \frac{105}{4}v_{0,xx}\right)\partial^3 + \left(\frac{105}{2}v_0v_{0,x} + \frac{175}{8}v_{0,xxx}\right)\partial^2 + \left(\frac{35}{2}v_0^3 + \frac{175}{4}v_0v_{0,xx} + \frac{245}{8}v_{0,x}^2 + \frac{161}{16}v_{0,xxxx}\right)\partial + \frac{63}{32}v_{0,xxxxx} + \frac{105}{8}v_0v_{0,xxx} + \frac{105}{4}v_{0,x}v_{0,xx} + \frac{105}{4}v_0^2v_{0,x}$$

$$\vdots, \tag{3.2}$$

Taking $B_m = B_0$, we list some evolution equations of the KdV hierarchy as follows:

- For the case of $B_n = B_3$, we have

$$\frac{\partial v_0}{\partial t_{03}} = \frac{1}{4}v_{0,xxx} + 3v_{0,x}v_0, \tag{3.3}$$

which is the well-known KdV equation. Under the scaling transformation $v_0 = \frac{1}{2}u$, $t_{03} = 4t$, (3.3) becomes the usual KdV equation.

- For the case of $B_n = B_5$, we have

$$\frac{\partial v_0}{\partial t_{05}} = \frac{1}{16}v_{0,xxxxx} + \frac{5}{4}v_0v_{0,xxx} + \frac{5}{2}v_{0,x}v_{0,xx} + \frac{15}{2}v_{0,x}v_0^2. \tag{3.4}$$

Under the scaling transformation $v_0 = \frac{1}{2}u$, $t_{05} = 16t$, (3.4) becomes the usual 5-order KdV equation.

- For the case of $B_n = B_7$, we have

$$\frac{\partial v_0}{\partial t_{07}} = \frac{1}{64}v_{0,xxxxxxx} + \frac{7}{16}v_0v_{0,xxxxx} + \frac{21}{16}v_{0,x}v_{0,xxxx} + \frac{35}{16}v_{0,xx}v_{0,xxx} + \frac{35}{2}v_0v_{0,x}v_{0,xx} + \frac{35}{8}v_0^2v_{0,xxx} + \frac{35}{8}v_{0,x}^3 + \frac{35}{2}v_{0,x}v_0^3. \tag{3.5}$$

Under the scaling transformation $v_0 = \frac{1}{2}u$, $t_{07} = 64t$, (3.5) becomes the usual 7-order KdV equation.

In the following, we will list some evolution equations of the generalized KdV hierarchy from the generalized Lax equation (2.4) except $B_m = B_0$. We also get the single soliton solution of some nonlinear evolution equations.

- Taking the operator pair (B_1, B_2) in (2.4), we have

$$\frac{\partial v_0}{\partial t_{12}} = -\frac{1}{4}v_{0,xxx} + v_0v_{0,x}. \tag{3.6}$$

Under the scaling transformation $v_0 = -\frac{3}{2}u$, $t_{12} = -4t$, (3.6) becomes the usual KdV equation.

- Taking the operator pair (B_1, B_3) in (2.4), we have

$$\frac{\partial v_0}{\partial t_{13}} = 0. \tag{3.7}$$

- Taking the operator pair (B_1, B_4) in (2.4), we have

$$\frac{\partial v_0}{\partial t_{14}} = -\frac{1}{16}v_{0,xxxxx} + \frac{11}{4}v_0v_{0,xxx} + \frac{7}{2}v_{0,x}v_{0,xx} + \frac{9}{2}v_{0,x}v_0^2. \tag{3.8}$$

Its single soliton solution is

$$v_0 = \frac{5(3\sec h^2\xi - 1)(5\sqrt{41} - 33)k^2}{-21 + 9\sqrt{41}}, \tag{3.9}$$

where $\xi = k(\omega t + x) + b$ in which

$$\omega = \frac{(-1019699 + 159231\sqrt{41})k^4}{-33606 + 5214\sqrt{41}},$$

b and k are arbitrary constants.

- Taking the operator pair (B_2, B_3) in (2.4), we have

$$\frac{\partial v_0}{\partial t_{23}} = \frac{1}{16}v_{0,xxxxx} + \frac{1}{2}v_0v_{0,xxx} + \frac{9}{4}v_{0,x}v_{0,xx} - 3v_{0,x}v_0^2. \tag{3.10}$$

Its single soliton solution is

$$v_0 = -\frac{5k^2}{2}(3\sec h^2\xi - 1), \tag{3.11}$$

where $\xi = k(\omega t + x) + b$ in which $\omega = -\frac{51k^4}{4}$, b and k are arbitrary constants.

- Taking the operator pairs (B_1, B_5) and (B_2, B_4) in (2.4), we have

$$\frac{\partial v_0}{\partial t_{15}} = \frac{\partial v_0}{\partial t_{24}} = 0. \tag{3.12}$$

- Taking the operator pair (B_1, B_6) in (2.4), we have

$$\begin{aligned}\frac{\partial v_0}{\partial t_{16}} = &-\frac{1}{64}v_{0,xxxxxxx} + \frac{57}{16}v_0v_{0,xxxxx} + \frac{119}{16}v_{0,x}v_{0,xxxx} + \frac{125}{16}v_{0,xx}v_{0,xxx}\\ &+\frac{35}{2}v_0v_{0,x}v_{0,xx} + \frac{45}{8}v_0^2v_{0,xxx} + \frac{25}{8}v_{0,x}^3 + \frac{25}{2}v_{0,x}v_0^3.\end{aligned} \tag{3.13}$$

- Taking the operator pair (B_2, B_5) in (2.4), we have

$$\begin{aligned}\frac{\partial v_0}{\partial t_{25}} = &\frac{1}{64}v_{0,xxxxxxx} + \frac{1}{4}v_0v_{0,xxxxx} + \frac{5}{4}v_{0,x}v_{0,xxxx} + \frac{35}{16}v_{0,xx}v_{0,xxx}\\ &+\frac{35}{4}v_0v_{0,x}v_{0,xx} + \frac{5}{8}v_0^2v_{0,xxx} + \frac{15}{4}v_{0,x}^3 - \frac{15}{2}v_{0,x}v_0^3.\end{aligned} \tag{3.14}$$

- Taking the operator pair (B_3, B_4) in (2.4), we have

$$\begin{aligned}\frac{\partial v_0}{\partial t_{34}} = &-\frac{1}{64}v_{0,xxxxxxx} + \frac{1}{2}v_0v_{0,xxxxx} - \frac{1}{4}v_{0,x}v_{0,xxxx} - \frac{27}{16}v_{0,xx}v_{0,xxx}\\ &+\frac{141}{4}v_0v_{0,x}v_{0,xx} + \frac{75}{8}v_0^2v_{0,xxx} + \frac{33}{4}v_{0,x}^3 + \frac{27}{2}v_{0,x}v_0^3.\end{aligned} \tag{3.15}$$

- Taking the operator pairs (B_1, B_7), (B_2, B_6) and (B_3, B_5) in (2.4), we have

$$\frac{\partial v_0}{\partial t_{17}} = \frac{\partial v_0}{\partial t_{26}} = \frac{\partial v_0}{\partial t_{35}} = 0. \tag{3.16}$$

From the above evolution equations, we can conjecture that when $m + n$ is even, the nonlinear evolution equation is $\frac{\partial v_0}{\partial t_{mn}} = 0$.

4. GENERALIZED BOUSSINESQ HIERARCHY

Equating the coefficients of the operator $\partial^{-i}(i = 1, 2, \cdots)$ in the constraints $(L^3)_- = 0$, we can derive

$$\begin{aligned}v_2 &= -v_0^2 - \frac{1}{3}v_{0,xx} - v_{1,x},\\ v_3 &= 2v_0v_{0,x} + \frac{1}{3}v_{0,xxx} - 2v_0v_1 + \frac{2}{3}v_{1,xx},\\ v_4 &= -v_0v_{0,xx} + \frac{5}{3}v_0^3 - v_{0,x}^2 - \frac{2}{9}v_{0,xxxx} + 4v_0v_{1,x} + 3v_1v_{0,x} - v_1^2 - \frac{1}{3}v_{1,xxx},\\ v_5 &= -10v_{0,x}v_0^2 + 5v_0^2v_1 + 5v_1v_{1,x} - \frac{20}{3}v_{0,x}v_{1,x} - \frac{10}{3}v_1v_{0,xx} - 5v_0v_{1,xx}\\ &\quad+\frac{1}{9}v_{0,xxxxx} + \frac{1}{9}v_{1,xxxx},\\ &\vdots\end{aligned} \tag{4.1}$$

Then we can derive B_n of the Boussinesq hierarchy are

$$\begin{aligned}B_1 &= \partial,\\ B_2 &= \partial^2 + 2v_0,\\ B_3 &= \partial^3 + 3v_0\partial + 3v_{0,x} + 3v_1,\\ B_4 &= \partial^4 + 4v_0\partial^2 + (4v_1 + 6v_{0,x})\partial + 2v_0^2 + \frac{8}{3}v_{0,xx} + 2v_{1,x},\\ B_5 &= \partial^5 + 5v_0\partial^3 + (5v_1 + 10v_{0,x})\partial^2 + (5v_0^2 + \frac{25}{3}v_{0,xx} + 5v_{1,x})\partial + 10v_0v_1\\ &\quad+\frac{10}{3}v_{1,xx} + 10v_0v_{0,x} + \frac{10}{3}v_{0,xxx},\end{aligned} \tag{4.2}$$

Similarly, when $m = 0$, we can list some evolution equations of the Boussinesq hierarchy as follows:

- For the case of $B_n = B_2$, we have

$$\begin{aligned} \frac{\partial v_0}{\partial t_{02}} &= v_{0,xx} + 2v_{1,x}, \\ \frac{\partial v_1}{\partial t_{02}} &= -\frac{2}{3}v_{0,xxx} - 2v_{0,x}v_0 - v_{1,xx}, \end{aligned} \tag{4.3}$$

  Eliminating v_1, replacing v_0 with $-u$, and replacing t_{02} with t, we can get

$$3\frac{\partial^2 u}{\partial t^2} + (u_{xxx} - 12uu_x)_x = 0, \tag{4.4}$$

  which is the well-known Boussinesq equation.
- For the case of $B_n = B_4$, we have

$$\begin{aligned} \frac{\partial v_0}{\partial t_{04}} &= \left(\frac{1}{3}v_{0,xxx} + 2v_0v_{0,x} + 4v_0v_1 + \frac{2}{3}v_{1,xx}\right)_x, \\ \frac{\partial v_1}{\partial t_{04}} &= -\frac{2}{9}v_{0,xxxxx} - 2v_0v_{0,xxx} - 4_{0,x}v_{0,xx} - 4v_0^2v_{0,x} - 2(v_0v_{1,x})_x - \frac{1}{3}v_{1,xxxx} + 4v_1v_{1,x}. \end{aligned} \tag{4.5}$$

  Under the scaling transformation $v_0 = \frac{1}{3}u, v_1 = -\frac{1}{3}v, t_{04} = -t$, (4.5) becomes the second equation of the Boussinesq hierarchy,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left(-\frac{1}{3}u_{xxx} - \frac{2}{3}uu_x + \frac{4}{3}uv + \frac{2}{3}v_{xx}\right)_x, \\ \frac{\partial v}{\partial t} &= -\frac{2}{9}u_{xxxxx} - \frac{2}{3}uu_{xxx} - \frac{4}{3}u_xu_{xx} - \frac{4}{9}u^2u_x + \frac{2}{3}(uv_x)_x + \frac{1}{3}v_{xxxx} + \frac{4}{3}vv_x. \end{aligned} \tag{4.6}$$

- For the case of $B_n = B_5$, we have

$$\begin{aligned} \frac{\partial v_0}{\partial t_{05}} &= 10v_1v_{1,x} - 5v_{0,x}v_0^2 + 5(v_{0,x}v_1)_x - \frac{5}{3}(v_0v_{0,xx})_x - \frac{1}{9}v_{0,xxxxx}, \\ \frac{\partial v_1}{\partial t_{05}} &= \left(-5v_1v_{1,x} - 5v_0^2v_1 - \frac{10}{3}(v_{0,x}v_1)_x - \frac{5}{3}v_0v_{1,xx} - \frac{1}{9}v_{1,xxxx}\right)_x. \end{aligned} \tag{4.7}$$

  Under the scaling transformation $v_0 = \frac{1}{3}u, v_1 = -\frac{1}{3}v, t_{05} = -t$, (4.7) becomes the third equation of the Boussinesq hierarchy,

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{10}{3}vv_x + \frac{5}{9}u_xu^2 + \frac{5}{3}(u_xv)_x + \frac{5}{9}(uu_{xx})_x + \frac{1}{9}u_{xxxxx}, \\ \frac{\partial v}{\partial t} &= \left(-\frac{5}{3}vv_x + \frac{5}{9}u^2v + \frac{10}{9}(u_xv)_x + \frac{5}{9}uv_{xx} + \frac{1}{9}v_{xxxx}\right)_x. \end{aligned} \tag{4.8}$$

  In the following, we will list some evolution equations of the generalized Boussinesq hierarchy from the generalized Lax equation (2.4) except $B_m = B_0$.
- Taking the operator pair (B_1, B_2) in (2.4), we have

$$\begin{aligned} \frac{\partial v_0}{\partial t_{12}} &= -\frac{1}{3}v_{0,xxx}, \\ \frac{\partial v_1}{\partial t_{12}} &= 4v_1v_{0,x} - 4v_0v_{1,x} - 2v_0v_{0,xx} + 2v_{0,x}^2 - \frac{1}{3}v_{1,xxx}. \end{aligned} \tag{4.9}$$

- Taking the operator pair (B_1, B_3) in (2.4), we have

$$\begin{aligned} \frac{\partial v_0}{\partial t_{13}} &= (2v_0v_1 + v_0v_{0,x} - \frac{2}{3}v_{1,xx} - \frac{1}{3}v_{0,xxx})_x, \\ \frac{\partial v_1}{\partial t_{13}} &= (2v_0v_{0,xx} + \frac{2}{9}v_{0,xxxx} + \frac{1}{3}v_{1,xxx} - v_0v_{1,x} + v_1^2 - \frac{2}{3}v_0^3)_x. \end{aligned} \tag{4.10}$$

- Taking the operator pair (B_1, B_4) in (2.4), we have

$$\begin{aligned} \frac{\partial v_0}{\partial t_{14}} &= \left(-v_0^3 + 3v_1^2 + 3v_1v_{0,x} + 3v_0v_{0,xx} + 2v_{0,x}^2\right)_x, \\ \frac{\partial v_1}{\partial t_{14}} &= -3(v_1v_{1,x})_x - 6v_1v_0v_{0,x} + 6v_1v_{0,xxx} - 4v_{0,xx}v_0^2 - 11v_{1,x}v_0^2 + \frac{7}{3}v_{1,xx}v_{0,x} \\ &\quad + 4v_{0,xx}v_{1,x} + \frac{5}{3}v_0v_{1,xxx} + \frac{2}{3}v_{0,x}v_{0,xxx} - \frac{2}{3}v_0v_{0,xxxx}. \end{aligned} \tag{4.11}$$

- Taking the operator pair (B_2, B_3) in (2.4), we have

$$\begin{aligned}\frac{\partial v_0}{\partial t_{23}} &= \left(-\frac{4}{3}v_0^3 + 2v_1^2 + 2v_1 v_{0,x} + v_0 v_{0,xx} + 2v_{0,x}^2 + \frac{1}{9}v_{0,xxxx}\right)_x,\\ \frac{\partial v_1}{\partial t_{23}} &= -2v_{1,x}^2 - 5v_1 v_{1,xx} + 2v_1 v_0 v_{0,x} - \frac{7}{3}v_1 v_{0,xxx} + 3v_{0,xx}v_0^2 + 2v_{1,x}v_0^2 + 4v_{1,xx}v_{0,x}\\ &\quad + v_{0,xx}v_{1,x} + 5v_0 v_{1,xxx} - \frac{2}{3}v_{0,x}v_{0,xxx} + 2v_0 v_{0,xxxx} + 2v_{0,x}^2 v_0 - v_{0,xx}^2 + \frac{1}{9}v_{1,xxxxx}.\end{aligned} \tag{4.12}$$

- Taking the operator pair (B_1, B_5) in (2.4), we have

$$\begin{aligned}\frac{\partial v_0}{\partial t_{15}} &= -\frac{1}{9}v_{0,xxxxxx} + \frac{20}{3}v_{0,xx}^2 - \frac{2}{9}v_{1,xxxxx} + \frac{20}{3}v_{0,x}v_{1,xx} + \frac{40}{3}v_{0,xx}v_{1,x}\\ &\quad + \frac{10}{3}v_0 v_{1,xxx} + 10v_1 v_{0,xxx} + \frac{25}{3}v_{0,xxx}v_{0,x} + \frac{5}{3}v_0 v_{0,xxxx},\\ \frac{\partial v_1}{\partial t_{15}} &= \frac{1}{9}v_{1,xxxxxx} + \frac{2}{27}v_{0,xxxxxxx} - \frac{100}{3}v_0 v_{0,x}v_{0,xx} - 20v_{1,x}v_1 v_0 - 10v_{1,x}v_{0,x}v_0\\ &\quad + \frac{10}{9}v_{0,xxxx}v_{0,x} - \frac{70}{9}v_{0,xxx}v_{0,xx} + \frac{4}{3}v_0 v_{0,xxxxx} - 10v_0^2 v_{0,xxx} + \frac{20}{3}v_{1,x}v_{1,xx}\\ &\quad + \frac{20}{3}v_0^3 v_{0,x} + 10v_1^2 v_{0,x} + 10v_1 v_{0,x}^2 - \frac{25}{3}v_{0,xxx}v_{1,x} + 10v_1 v_{1,xxx}\\ &\quad - \frac{5}{3}v_0 v_{1,xxxx} - \frac{20}{3}v_{1,xx}v_{0,xx} - \frac{10}{3}v_{0,x}^3 - 10v_{0,xx}v_1 v_0.\end{aligned} \tag{4.13}$$

- Taking the operator pair (B_2, B_4) in (2.4), we have

$$\begin{aligned}\frac{\partial v_0}{\partial t_{24}} &= \frac{1}{9}v_{0,xxxxxx} + 3v_{0,xx}^2 + \frac{2}{9}v_{1,xxxxx} + 13v_{0,x}v_{1,xx} + 6v_{0,xx}v_{1,x} + \frac{22}{3}v_0 v_{1,xxx} + \frac{1}{3}v_1 v_{0,xxx}\\ &\quad + \frac{20}{3}v_{0,xxx}v_{0,x} + \frac{11}{3}v_0 v_{0,xxxx} - 12v_1 v_{0,x}v_0 - 3v_{0,xx}v_0^2 - 6v_{1,x}v_0^2 - 6v_{0,x}^2 v_0,\\ \frac{\partial v_1}{\partial t_{24}} &= -\frac{1}{9}v_{1,xxxxxx} - \frac{2}{27}v_{0,xxxxxxx} - 22v_0 v_{0,x}v_{0,xx} - 12v_{1,x}v_1 v_0 + 6v_{1,x}v_{0,x}v_0\\ &\quad - \frac{10}{3}v_{0,xxxx}v_{0,x} - \frac{8}{3}v_{0,xxx}v_{0,xx} - \frac{8}{3}v_0 v_{0,xxxxx} - \frac{16}{3}v_0^2 v_{0,xxx} + \frac{71}{3}v_{1,x}v_{1,xx}\\ &\quad + 6v_0^3 v_{0,x} - 16v_1^2 v_{0,x} - 10v_1 v_{0,x}^2 + \frac{26}{3}v_{0,xxx}v_{1,x} + \frac{41}{3}v_1 v_{1,xxx} - \frac{11}{3}v_0 v_{1,xxxx}\\ &\quad + \frac{7}{3}v_{1,xx}v_{0,xx} - \frac{20}{3}v_{0,x}^3 + 3v_{1,xx}v_0^2 - \frac{10}{3}v_{1,xxx}v_{0,x} + \frac{20}{3}v_1 v_{0,xxxx}.\end{aligned} \tag{4.14}$$

5. SUMMARY

In this paper, in terms of the Lax triple (B_m, B_n, L), we investigated the generalized Lax equation of the KdV and Boussinesq hierarchies. When $m = 0$, the generalized Lax equation reduces to the usual Lax equation. We derived integrable evolution equations from the KdV and Boussinesq hierarchies. We got some soliton wave solutions from the nonlinear evolution equations of the generalized KdV hierarchy. Moreover, the evolution equations for the generalized KdV hierarchy seemed to be $\frac{\partial v_0}{\partial t_{mn}} = 0$ when $m + n$ is even. We also derived some generalized nonlinear evolution equations from the generalized Boussinesq Lax equation. More properties of the generalized KdV and Boussinesq hierarchies still deserve further study.

CONFLICTS OF INTEREST

The authors declare they have no conflicts interest.

AUTHORS' CONTRIBUTION

All authors completed the paper together. All authors read and approved the final manuscript.

FUNDING

This work is partially supported by National Natural Science Foundation of China (Grant No. 11801292) and the Fundamental Research Funds for the Central Universities (DUT19LK26).

ACKNOWLEDGMENTS

We thank the valuable suggestions of the referees.

REFERENCES

[1] C.H. Chen, K. Furuuchi, P.M. Ho, T. Takimi, More on the Nambu-Poisson M5-brane theory: scaling limit, background independence and an all order solution to the Seiberg-Witten map, J. High Energ. Phys. 10 (2010), 100.
[2] M.R. Chen, S.K. Wang, K. Wu, W.Z. Zhao, Infinite-dimensional 3-algebra and integrable system, J. High Energ. Phys. 12 (2012), 30.
[3] M.R. Chen, S.K. Wang, X.L. Wang, K. Wu, W.Z. Zhao, On $W_{1+\infty}$ 3-algebra and integrable system, Nucl. Phys. B 891 (2015), 655–675.
[4] M.R. Chen, Y. Chen, Z.W. Yan, J.Q. Mei, X.L. Wang, A 3-Lie algebra and the dKP hierarchy, J. Nonlinear Math. Phys. 26 (2019), 91–97.
[5] E. Date, M. Kashiwara, M. Jimbo, T. Miwa, Transformation groups for soliton equations, in: M. Jimbo, T. Miwa (Eds.), Proceedings of RIMS symposium on Non-linear Integrable Systems-Classical and Quantum Theory, World Scientific, Singapore, 1983, pp. 39–119.
[6] L.A. Dickey, Soliton equations and Hamiltonian Systems, 2nd ed., World Scintific, Singapore, 2003, p. 420.
[7] M.H. Li, J.P. Cheng, C.Z. Li, J.S. He, The recursion operators of the BKP hierarchy and the CKP hierarchy, Sci. Sin. Math. 43 (2012), 499–514 (in Chinese).
[8] Y. Nambu, Generalized Hamiltonian dynamics, Phys. Rev. D. 7 (1973), 2405–2412.
[9] G. Papadopoulos, M2-branes, 3-Lie algebras and Plücker relations, J. High Energ. Phys. 05 (2008), 054.
[10] L. Takhtajan, On foundation of the generalized Nambu mechanics, Comm. Math. Phys. 160 (1994), 295–315.
[11] X.L. Wang, J.Q. Mei, M.L. Li, Z.W. Yan, On generalized Lax equations of the Lax triple of the BKP and CKP hierarchies, J. Nonlinear Math. Phys. 24 (2017), 171–182.
[12] X.L. Wang, L. Yu, Y.X. Yang, M.R. Chen, On generalized Lax equation of the Lax triple of KP hierarchy, J. Nonlinear Math. Phys 22 (2015), 194–203.
[13] H.X. Wu, X.J. Liu, Y.B. Zeng, Two new multi-component BKP hierarchies, Commun. Theor. Phys. 51 (2009), 193–199.
[14] Y.X. Yang, S.K. Yao, C.H. Zhang, W.Z. Zhao, Realization of the infinite-dimensional 3-algebras in the Calogero–Moser model, Chin. Phys. Lett. 32 (2015), 040202.